Multiband Robust Optimization: theory and applications

Fabio D’Andrea Agiovanni
Something about me

Fundamentals of Robust Optimization

A classic: the Bertsimas-Sim model

Multiband Uncertainty in Robust Optimization

An application: Wireless Network Design

All the presented results are strongly based on discussions with experts from our industrial partners, such as:

agcom  BT  DFN  Nokia Siemens Networks  Space Engineering

and are based on realistic network data. The network models were validated by the Partners, as well.
## Education and experience

### EDUCATION

- 2004: Bachelor of Science in Industrial Engineering
- 2006: Master of Science in Industrial Engineering
- 2010: Ph.D. in Operations Research

### PROFESSIONAL EXPERIENCE

- **2006 - 2009:** Research Fellow, Sapienza Università di Roma
- **2008 - 2009:** Research Scholar, Columbia University
- **2009 - 2010:** Post-doc, Sapienza Università di Roma

**Increasing responsibilities** in the Berlin Mathematical Research Community

- **2010 - 2011:** Post-doc, Zuse Institute Berlin
- **2011 - 2015:** Senior Researcher, Technical University Berlin and Zuse Institute Berlin
- **2014 - ongoing:** Project Director, Einstein Center for Mathematics
- **From 10-2015:** Head of Research Group, Zuse Institute Berlin
- **From 10-2015:** Lecturer, Technical University Berlin and Freie Universität Berlin
Research: main topics

Theory and applications of:

- Mixed Integer Linear Programming
  - Polyhedral analysis (strong formulations)
  - Cutting-plane methods

- Optimization under Data Uncertainty
  - Robust Optimization
  - Cardinality-constrained uncertainty sets

- Capacitated Network Design
  - (Strong) valid inequalities characterization
  - Efficient flow-routing algorithms
MY AIM: bridging the gap between optimization theory and practice

- **Wireless Network Design**
  - User service coverage with quality-of-service guarantees
  - Robustness against signal propagation uncertainty

- **Optical Fiber Network Design**
  - Capacity and data routing design
  - Robustness against traffic uncertainty and failures

- **Power System Optimization**
  - Unit Commitment
  - Robust energy offering under price uncertainty

+ Many other **math-in-industry research** and **consulting projects** for/with e.g.
It’s an uncertain world

Most real-world optimization problems involve **uncertain data**

For each datum, we know a **reference value** that however **generally differs from the actual value**

Some causes:
- errors in measurements
- estimations from historical data
- finite numerical representation of computers

Some examples:
- **Train Scheduling** (delays)
- **Wireless Networks** (signal propagation)
- **Power Systems** (market price)
- **Surgery Scheduling** (requests of operations)
Data uncertainty in Optimization

**CLASSIC OPTIMIZATION**

\[
\begin{align*}
\text{max} & \quad c^t x \\
A x & \leq b \\
x & \geq 0^n
\end{align*}
\]

**THE VALUE OF ALL COEFFICIENTS IS KNOWN EXACTLY**

**REASONABLE ASSUMPTION FOR ANY PROBLEM? NO!**

Neglecting data uncertainty may lead to bad surprises:

- nominal optimal solutions may result heavily suboptimal
- nominal feasible solutions may result infeasible

To avoid such situations, we want to find **robust solutions**:

**ROBUST SOLUTION** = solution that remains feasible even when the input data vary (PROTECTION AGAINST DATA DEVIATIONS)

**THEY OVERLOOKED DATA UNCERTAINTY...**
A simple numerical example may clarify the effects of data deviations:

Suppose that we have computed an optimal solution $x=1, y=1$ for some problem with nominal constraint:

$$100x + 200y \leq 300$$

However, we have neglected that the coefficient of $x$ may deviate up to 10%, so we could have

$$(100 + 10) + 200 > 300$$

What if this was part of a problem to detect water contamination?
An example: traffic uncertainty in Network Design

- In every origin-destination pair, traffic volume heavily fluctuates over the week.

- Overall fluctuation in a network link even more severe.

- Solution of the professional: dimension network capacity by (greatly) overestimating demand.

? CAN WE DEFINE A BETTER ROBUST SOLUTION THROUGH OPTIMIZATION? ?
Robust Optimization

Data uncertainty is modeled as hard constraints that restrict the feasible set
[Ben-Tal, Nemirovski 98, El-Ghaoui et. al. 97]

\[
\begin{align*}
\text{NOMINAL PROBLEM} & \quad \text{Coefficients are uncertain!!!} \quad \text{ROBUST COUNTERPART} \\
\max \ & c'x \\
A \ x & \leq b \\
x & \geq 0^n \\
\end{align*}
\]

\[
\begin{align*}
a_{ij} & = \bar{a}_{ij} + \delta_{ij} \\
\tilde{A} \ x & \leq b \quad \tilde{A} \in \mathcal{A} \\
x & \geq 0^n \\
\end{align*}
\]

- \( \mathcal{A} \) should reflect the risk aversion of the decision maker
- protection entails the so-called Price of Robustness
The Bertsimas-Sim model

\[
\begin{align*}
\max & \quad \sum_{j \in J} c_j x_j \\
\sum_{j \in J} a_{ij} x_j & \leq b_i \quad i \in I = \{1, \ldots, m\} \\
x_j & \geq 0 \quad j \in J = \{1, \ldots, n\}
\end{align*}
\]

Assumptions:

1) w.l.o.g. uncertainty just affects the coefficient matrix

2) the coefficients are independent random variables following an unknown symmetric distribution over a symmetric range

Deviation range: each coefficient \( a_{ij} \) assumes value in the symmetric range \( a_{ij} \in [\tilde{a}_{ij} - a_{ij}^{\max}, \tilde{a}_{ij} + a_{ij}^{\max}] \)

Row-wise uncertainty: for each constraint \( i \), \( \Gamma_i \in [0, n] \) specifies the max number of coefficients deviating from \( a_{ij} \)

\[
\begin{align*}
\max & \quad \sum_{j \in J} c_j x_j \\
\sum_{j \in J} \tilde{a}_{ij} x_j + \text{DEV}(x, \Gamma_i) & \leq b_i \quad \forall i \in I \\
x_j & \geq 0 \quad \forall j \in J
\end{align*}
\]

ROBUST COUNTERPART (NON-LINEAR)

ROBUST COUNTERPART [Bertsimas, Sim 04] (LINEAR AND COMPACT)

\[
\begin{align*}
\max & \quad \sum_{j \in J} c_j x_j \\
\sum_{j \in J} \tilde{a}_{ij} x_j + \Gamma_i w_i + \sum_{j \in J} z_{ij} & \leq b_i \quad \forall i \in I \\
\begin{align*}
w_i + z_{ij} & \geq a_{ij}^{\max} j x_j \\
z_{ij} & \geq 0 \\
w_i & \geq 0 \\
x_j & \geq 0
\end{align*} \quad \forall i \in I, j \in J
\end{align*}
\]
Using the BS model in practice

- In real-world problems, historical data about the deviations of the uncertain coefficients are commonly available.
- The data can be easily used to build histograms representing the distribution of the deviations.

ARE WE REALLY ABLE TO EXPLOIT SUCH INFORMATION WITH THE BERTSIMAS-SIM MODEL?

The behaviour of the uncertainty internally to the deviation range is completely neglected (focus on the extreme deviations).

According to our past experiences, practitioners would definitely prefer a more refined representation of the uncertainty.

Example: no. of coefficients deviating between [+40,+50]% from the nominal value.
What can we do to increase our modeling capacity?

Adopt a multi-band uncertainty set

Histogram of observed deviations

Example: no. of coefficients deviating between [+40,+50]% from the nominal value

Strongly data-driven uncertainty set

A general theoretical study was missing!

Our aim has been to fill such gap
Formalizing Multiband Uncertainty

Focus on the coefficients $a_{ij}$ of each constraint $i$ (row-wise uncertainty)

K deviation values $-\infty < d_{ij}^{i-1} < \cdots < d_{ij}^{0} = 0 < \cdots < d_{ij}^{K+} < +\infty$ for each coefficient $a_{ij}$

K deviation bands such that each band $k$ corresponds with range $(d_{ij}^{k-1}, d_{ij}^{k})$

Lower and upper bounds $0 \leq l_{k} \leq u_{k} \leq n$ on the number of coefficients deviating in each band $k$

No upper bound on band $k = 0$, i.e. $u_{0} = n$

There exists a feasible assignment $\sum_{k \in K} l_{k} \leq n$
Focus on the coefficients $a_{ij}$ of each constraint $i$ (row uncertainty)

For each coefficient $a_{ij}$, we have a number of past observations $\hat{a}_{ij}$

Compute the percentage deviation of an observation from the nominal value $\frac{\hat{a}_{ij} - \bar{a}_{ij}}{\bar{a}_{ij}} \cdot 100$

Build the histogram representing the distribution of the percentage deviations for the considered constraint

Example

OBSERVED DISCRETE DISTRIBUTION
(ALL COEFFICIENTS IN THE CONSTRAINT)

POSSIBLE MULTI-BAND SET FOR THE CONSTRAINT
(assuming 100 coefficients in the constraint)

$U_{-1} = 33$

$L_{-1} = 27$

$+/− 10\% \text{ OF THE EXPECTED NUMBER OF COEFFICIENTS FALLING IN EACH BAND OF THE HISTOGRAM}$
The max-deviation auxiliary problem under MB

**MILP**

\[
\begin{align*}
\text{max} & \sum_{j \in J} c_j x_j \\
\sum_{j \in J} \bar{a}_{ij} x_j & \leq b_i \quad \forall i \in I \\
x_j & \geq 0 \\
x_j & \in \mathbb{Z}_+
\end{align*}
\]

\nonumber

**DEV01**

\[
\begin{align*}
\text{max} & \sum_{j \in J} \sum_{k \in K} d_{ij}^k x_j y_{ij}^k \\
\sum_{j \in J} y_{ij}^k & \leq u_k \quad k \in K \\
\sum_{k \in K} y_{ij}^k & \leq 1 \quad j \in J \\
y_{ij}^k & \in \{0, 1\} \quad j \in J, k \in K
\end{align*}
\]

**MAXIMIZATION OF TOTAL DEVIATION**

**BOUNDS ON THE NO. OF COEFFICIENTS FALLING IN BAND k**

**EACH COEFFICIENT FALLS IN AT MOST ONE BAND**
PROPOSITION 1 (Büsing & D’Andreagiovanni 12)

The polytope associated with (DEV01) is integral.

Proof based on showing that the coefficient matrix of (DEV01) is totally unimodular

THEOREM 1 (Büsing & D’Andreagiovanni 12)

The Robust Counterpart of (MILP) under multi-band uncertainty is equivalent to:

\[
\begin{align*}
& \text{max } \sum_{j \in J} c_j x_j \\
& \sum_{j \in J} \bar{a}_{ij} x_j - \sum_{k \in K} l_k v_i^k + \sum_{k \in K} w_k w_i^k + \sum_{j \in J} z_j^i \leq b_i & i \in I \\
& -v_i^k + w_i^k + z_j^i \geq d_{ij}^k x_j & i \in I, j \in J, k \in K \\
& v_i^k, w_i^k \geq 0 & i \in I, k \in K \\
& z_j^i \geq 0 & i \in I, j \in J \\
& x_j \geq 0 & j \in J \\
& x_j \in \mathbb{Z}_+ & j \in J_Z \subseteq J
\end{align*}
\]

Proof based on exploiting the integrality of (DEV01) and strong duality

If the original problem is linear, then also the counterpart is linear
**GOAL:** finding a robust optimal solution for multi-band set $D$ through a cutting-plane algorithm

**Separation problem**

Given a solution $x \in \mathbb{R}_+^{n-\lambda} \times \mathbb{Z}_+^\lambda$, is this solution **robust feasible** for constraint $i$?

$$x \in \mathbb{R}_+^{n-\lambda} \times \mathbb{Z}_+^\lambda \quad \text{robust feasible for } i \iff \sum_{j \in J} \tilde{a}_{ij} x_j + DEV_i(x, D) \leq b_i$$

If this condition does not hold and $y^*$ is an optimal solution to (DEV01) then

$$\sum_{j \in J} \tilde{a}_{ij} x_j + \sum_{j \in J} \sum_{k \in K} d^k_{ij} x_j y^*_{ij} \leq b_i$$

is a valid inequality for the original formulation and cuts off $x$ (**robustness cut**)

**THEOREM 2** (Büsing & D'Andréa-Giovanni 12)

Separating a robustness cut corresponds with solving a min-cost flow problem

Proof based on showing the 1:1 correspondence between integral flows and assignments $y$ of (DEV01)
Solving (DEV01) is equivalent to solving a min-cost flow problem on the following graph (B. & D’A.12)

**SET OF NODES**
- one node \(v_j\) for each variable \(x_j\)
- one node \(w_k\) for each band \(K\)
- source and sink

**SET OF EDGES**
(each associated with a triple (flow LB, flow UB, unitary cost))
- one edge \((s, v_j)\) with triple \((0,1,0)\) for each \(x_j\)
- one edge \((v_j, w_k)\) with triple \((0,1, -d_{ij}^k x_j)\) for each \(x_j\) and \(k\)
- one edge \((w_k, t)\) with triple \((l_k, u_k,0)\) for each \(k\)

**PROPERTIES:**
- 1:1 correspondence between integral flows of value \(n\) and complete assignments \(y\) of (DEV01)
- The cost relation between corresponding integral flow and assignment is: \(d(x,y) = -c(f)\)
- The following chain of equalities holds:

\[
DEV_i(x, D) = \max_{y_i \in \bar{Y}_i} d(x, y_i) = -\min_{y_i \in \bar{Y}_i} -d(x, y_i) = -\min_{f \in F_i} c(f) = -c^*(x)
\]
Basic robust cutting plane algorithm

1. **Problem Data**
   - Does a violated robustness cut exist?

2. **Define the Nominal Problem (MILP)**
   - Solve (MILP)

3. **Find the maximum deviation allowed by the multi-band set for each constraint of (MILP)**

4. **Add the robustness cut to (MILP)**

   - **Yes**: Does a violated robustness cut exist?
   - **No**: Optimal & robust solution

*Fabio D'Andreagiovanni – Multiband Robust Optimization*
0-1 Programs with Multiband cost uncertainty

\[
\begin{align*}
\min \ c' x \\
x \in X \subseteq \{0, 1\}^n \\
\end{align*}
\]

\[
\begin{align*}
\min \ & \sum_{j \in J} c_j x_j + \sum_{k \in K} \theta_k w_k + \sum_{j \in J} z_j \\
\ & w_k + z_j \geq d^k_j x_j \\
\ & w_k \geq 0 \\
\ & z_j \geq 0 \\
\ & x \in X,
\end{align*}
\]

THEOREM (Büsing & D’Andreagiovanni 12)

The robust optimal solution can be obtained by solving a polynomial number of nominal problems with modified cost coefficients. Tractability and approximability of the algorithm used to solve the nominal problem are preserved.

NOT JUST a trivial extension of the Bertsimas-Sim results and proofs!
Dominance among multiband uncertainty sets
(great reduction in the compact robust counterpart size)
(Büsing & D’Andreagiovanni 2012)

Probability bounds of constraint violation
(Büsing & D’Andreagiovanni 2012)

(Strong) valid inequalities for 0-1 Linear Programs
(D’Andreagiovanni & Raymond, 2013)

Robust cutting planes for 0-1 linear programs with correlated uncertain right-hand-sides
(D’Andreagiovanni, 2014)
Comparing Gamma and Multiband Robustness

- Multiband linearity requires: \(( m \ n ) K\) additional constraints
  \(( m + m \ n ) K\) additional variables

- In the general case, MB can be less or more conservative than BS, depending upon the multiband structure (in our computational experience MB was always less conservative, when using realistic multiband sets and comparing them with realistic and fair Gamma-parameter)

- Anyway, we can derive some sufficient conditions for MB to be less conservative
  (by using majorizations/minorizations that however reduces the actual advantage of MB over BS)

\[
\sum_{j=1}^{\Gamma} \left( 1 - \frac{k[j]}{K} \right) - \sum_{j=\Gamma+1}^{\sum_{k \in K} + \theta_k} \frac{k[j]}{K} \geq 0
\]

Remarks:

- The condition is independent from the solution \(x\)

- Useful condition to check that “rational” histogram representations of major distributions like the exponential and the normal ensures that MB is less conservative than BS
Multiband Robustness - applications

ROBUST POWER ASSIGNMENT IN WIRELESS NETWORKS (fading uncertainty)

ROBUST TIME-OFFSET SETTING IN DVB-T NETWORKS (time propagation uncertainty)

ROBUST WIRELESS NETWORK JAMMING (jamming uncertainty)

ROBUST HARVEST SCHEDULING (timber production uncertainty)

MULTIPERIOD CAPACITATED NETWORK DESIGN (traffic uncertainty)

ENERGY OFFERING FOR A PRICE-TAKER (price uncertainty)
Reduced price of robustness w.r.t. what you have seen at AIRO 2015
A **Wireless Network** can be essentially described as a *set of transmitters* $T$ which provide for a telecommunication service to a *set of receivers* $R$ located in a target area.

Every transmitter is characterized by a *set of parameters*.

**Positional** (antenna height, geographical location)

**Radio-electrical** (e.g., power emission, frequency channel)

**WIRELESS NETWORK DESIGN PROBLEM (WND)**

set the values of the parameters of each transmitter to maximize a profit function, while ensuring a minimum quality of service for each served receiver.
Every receiver \( r \) picks up signals from all the transmitters, BUT:
- coverage is provided by a single transmitter, chosen as server of \( r \)
- all the other transmitters interfere the serving signal

If we introduce a continuous variable \( 0 \leq p_t \leq P_{\text{max}} \) to represent power emission of transmitter \( t \), \( r \) is covered if the signal-to-interference ratio (SIR) is higher than a given threshold:

\[
\frac{a_{r\sigma} \cdot p_\sigma}{\sum_{t \in T \setminus \{\sigma\}} a_{rt} \cdot p_t} \geq \delta
\]

(SIR constraint)
A fading coefficient $a_{rt}$ is usually computed through a propagation model and depends on several factors such as:

- the distance between $t$ and $r$
- the presence of obstacles
- the weather

The fading coefficients are naturally subject to **uncertainty**. Neglecting uncertainty may lead to plans with **unexpected coverage holes**.
Robust Power Assignment Problem

POWER ASSIGNMENT PROBLEM (PAP)

set the power emission of each transmitter to provide coverage to a set of receivers while minimizing the total power emission

\[
\begin{align*}
\min & \quad \sum_{t \in T} p_t \\
\text{subject to} & \quad a_{r\sigma} \cdot p_{\sigma} - \delta \sum_{t \in T \setminus \{\sigma\}} a_{rt} \cdot p_t - \text{DEV}(a,p) \geq 0 \quad r \in R, \sigma \in T \\
& \quad 0 \leq p_t \leq P^{\text{max}} \quad t \in T
\end{align*}
\]

POWER MINIMIZATION

SIR CONSTRAINTS

POWER BOUNDS

We take into account fading uncertainty by subtracting the worst power deviation in the l.h.s. of each SIR constraint

To solve this robust problem we can adopt **multiband robustness** and either:

- solve its linear and robust counterpart
- find a robust optimal solution by the robust cutting-plane approach
**Computational experience**

**TEST-BED:** 15 WiMAX instances with up to 180 transmitters and 2118 testpoints defined in collaboration with wireless network professionals

- fading coefficients assumed to be independent log-normal random variables (ITU Recommendation)
- 5 deviations bands (2 negative, 2 positive)
- all instances solved within one hour (Cplex 12.1, 4GB RAM)

<table>
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<th>NO. CONSTRAINTS AND VARIABLES (nominal problem)</th>
<th>NO. CONSTRAINTS AND VARIABLES (compact robust counterpart)</th>
<th>PRICE OF ROBUSTNESS % (Gamma-Robustness)</th>
<th>REDUCTION IN THE PRICE OF ROBUSTNESS % (Multiband Robustness wrt Gamma-Robustness)</th>
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Concluding remarks

- World is stochastic and most of real-world optimization problems involve uncertain data
- Robust Optimization is a modern and effective paradigm for dealing with data uncertainty
- We introduced Multiband Robust Optimization to generalize and refine the Bertsimas-Sim model

FUNDAMENTAL RESULTS:
- compact robust counterpart (purely linear if the nominal problem is purely Linear)
- efficient separation of robustness cuts by min-cost flow
- experiments on real-world problems indicate a sensible reduction in the price of robustness

ESSENTIAL REFERENCES:

C. Büsing, F. D’Andreagiovanni
New results about multi-band Uncertainty in Robust Optimization, Proc. of SEA 2012, LNCS 7276, 63-74

T. Bauschert, C. Büsing, F. D’Andreagiovanni, A. Koster, M. Kutschka, U. Steglich,
Network planning under demand uncertainty with Robust Optimization, IEEE Communications Magazine 52(2), 178-185, 2014

F. D’Andreagiovanni, C. Mannino, A. Sassano
GUB Covers and Power-Indexed formulations for Wireless Network Design,
Management Science 59(1), 142-156, 2013 INFORMS Telecom Best Paper Award 2014