Multiband Robust Optimization: theory and applications

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Presentation outline

- Something about me
- Fundaments of Robust Optimization
- A classic: the Bertsimas-Sim model
- Multiband Uncertainty in Robust Optimization
- An application: Wireless Network Design

All the presented results are strongly based on discussions with experts from our industrial partners, such as :

and are based on realistic network data. The network models were validated by the Partners, as well.

Education and experience

EDUCATION

- 2004: Bachelor of Science in Industrial Engineering
- 2006: Master of Science in Industrial Engineering
- 2010: Ph.D. in Operations Research

PROFESSIONAL EXPERIENCE

- 2006 - 2009: **Research Fellow**, Sapienza Università di Roma
- 2008 - 2009: **Research Scholar**, Columbia University
- 2009 - 2010: **Post-doc**, Sapienza Università di Roma

Increasing responsibilities in the Berlin Mathematical Research Community

- 2010 - 2011: **Post-doc**, Zuse Institute Berlin
- 2011- 2015: **Senior Researcher**, Technical University Berlin and Zuse Institute Berlin
- 2014 - ongoing: **Project Director**, Einstein Center for Mathematics
- From 10-2015: **Head of Research Group**, Zuse Institute Berlin
- From 10-2015: **Lecturer**, Technical University Berlin and Freie Universität Berlin

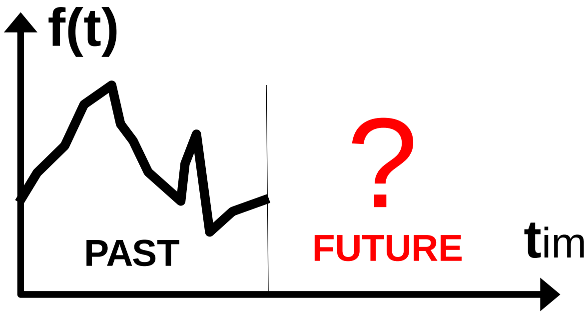
Research: main topics

Theory and applications of:

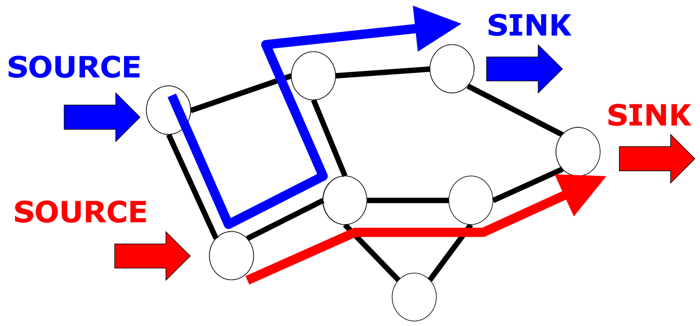
- Mixed Integer Linear Programming
  - Polyhedral analysis (strong formulations)
  - Cutting-plane methods

$$\begin{aligned} \max \quad & c'\,x \\ & A\,x \leq b \\ & x \in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \end{aligned}$$

- Optimization under Data Uncertainty
  - Robust Optimization
  - Cardinality-constrained uncertainty sets



- Capacitated Network Design
  - (Strong) valid inequalities characterization
  - Efficient flow-routing algorithms



Research: Real-world optimization

MY AIM: bridging the gap between optimization theory and practice

- **Wireless Network Design**
  - User service coverage with quality-of-service guarantees
  - Robustness against signal propagation uncertainty

- **Optical Fiber Network Design**
  - Capacity and data routing design
  - Robustness against traffic uncertainty and failures

- **Power System Optimization**
  - Unit Commitment
  - Robust energy offering under price uncertainty

A MAJOR EUROPEAN ELECTRIC UTILITY
CONFIDENTIAL

\+ many other **math-in-industry research** and **consulting projects** for/with e.g.

It's an uncertain world

Most real-world optimization problems involve uncertain data

For each datum, we know a **reference value that** however **generally differs from the actual value**

Some causes:

- errors in measurements
- estimations from historical data
- finite numerical representation of computers

Some examples:

Train Scheduling
(delays)

Wireless Networks
(signal propagation)

Power Systems
(market price)

Surgery Scheduling
(requests of operations)

Data uncertainty in Optimization

CLASSIC OPTIMIZATION

$$\begin{aligned} \max \quad & c' \, x \\ & A \, x \le b \\ & x \ge 0^n \end{aligned}$$

THE VALUE OF ALL COEFFICIENTS **IS KNOWN EXACTLY**

REASONABLE ASSUMPTION FOR ANY PROBLEM ? NO!

Neglecting data uncertainty may lead to bad surprises:

- nominal optimal solutions may result heavily suboptimal
- nominal feasible solutions may result infeasible

THEY OVERLOOKED DATA UNCERTAINTY…

To avoid such situations, we want to find **robust solutions:**

ROBUST SOLUTION = solution that remains feasible even when the input data vary (**PROTECTION AGAINST DATA DEVIATIONS**)

It was not robust...

- A simple numerical example may clarify the effects of data deviations:

Suppose that we have computed an optimal solution x=1, y=1 for some problem with nominal constraint:

$$100\,x + 200\,y \leq 300$$

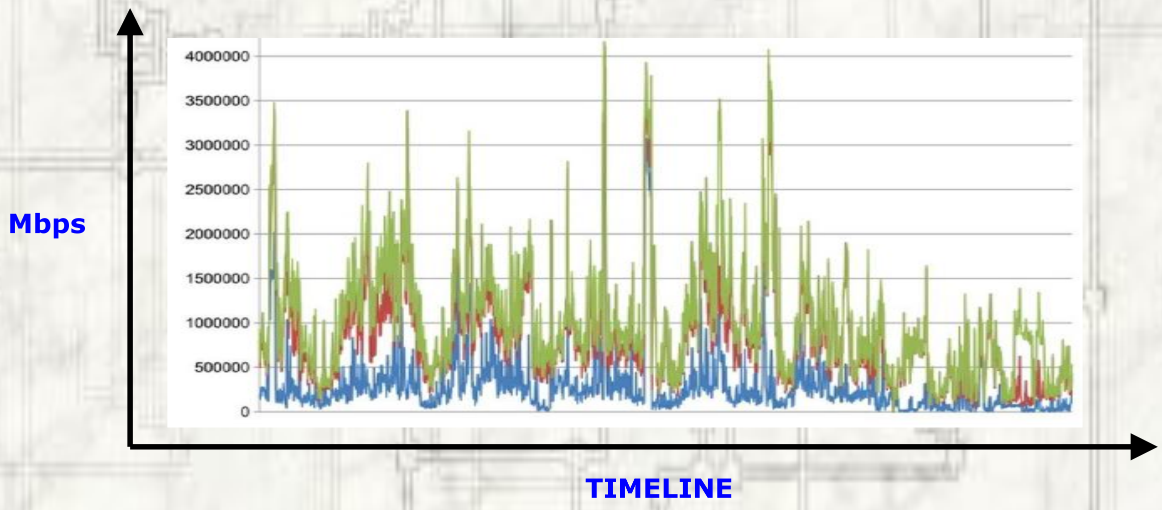
However, we have neglected that the coefficient of *x* may deviate up to 10%, so we could have

$$(100 + 10) + 200 > 300$$

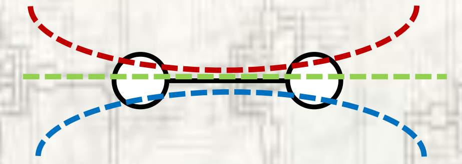
OPTIMAL SOLUTION ACTUALLY INFEASIBLE!

- What if this was part of a problem to detect water contamination?

An example: traffic uncertainty in Network Design



Traffic fluctuations of three O-D pairs
in the USA Abilene Network
(one-week observation)



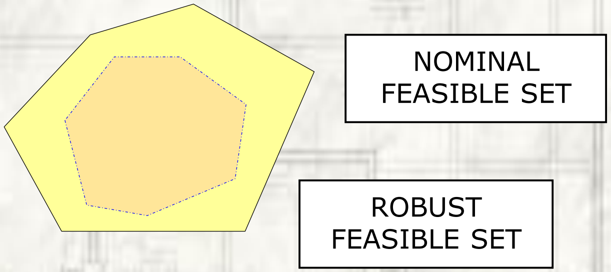
- In every origin-destination pair, traffic volume heavily fluctuates over the week
- Overall fluctuation in a network link even more severe
- Solution of the professional: dimension network capacity by **(greatly) overestimating demand**

? CAN WE DEFINE A BETTER ROBUST SOLUTION THROUGH OPTIMIZATION ?

Robust Optimization

Data uncertainty is modeled as hard constraints that restrict the feasible set

[Ben-Tal, Nemirovski 98, El-Ghaoui et. al. 97]



NOMINAL PROBLEM → Coefficients are uncertain!!! → ROBUST COUNTERPART

$$\max \; c' \, x$$
$$A \, x \leq b$$
$$x \geq 0^n$$

$$a_{ij} = \bar{a}_{ij} + \delta_{ij}$$

ACTUAL VALUE | NOMINAL VALUE | DEVIATION

$$\max \quad c' \, x$$
$$\tilde{A} \, x \leq b \quad \tilde{A} \in \mathcal{A}$$
$$x \geq 0^n$$

- $\mathcal{A}$ should reflect the risk aversion of the decision maker
- protection entails the so-called Price of Robustness

The Bertsimas-Sim model

$$\max \sum_{j \in J} c_j \, x_j$$
$$\sum_{j \in J} a_{ij} \, x_j \le b_i \qquad i \in I = \{1, \dots, m\}$$
$$x_j \ge 0 \qquad j \in J = \{1, \dots, n\}$$

Assumptions:

1) w.l.o.g. uncertainty just affects the coefficient matrix

2) the coefficients are independent random variables following an unknown **symmetric distribution over a symmetric range**

Deviation range: each coefficient a_{ij} assumes value in the symmetric range $a_{ij} \in [\bar{a}_{ij} - d_{ij}^{\max}, \bar{a}_{ij} + d_{ij}^{\max}]$

Row-wise uncertainty: for each constraint *i*, $\Gamma_i \in [0, n]$ specifies the max number of coefficients deviating from a_{ij}

ROBUST COUNTERPART (NON-LINEAR) → ROBUST COUNTERPART [Bertsimas, Sim 04] (LINEAR AND COMPACT)

ROBUST COUNTERPART (NON-LINEAR)

$$\max \sum_{j \in J} c_j x_j$$
$$\sum_{j \in J} \bar{a}_{ij} x_j \boxed{+ DEV(x, \Gamma_i)} \le b_i \qquad \forall i \in I$$
$$x_j \ge 0 \qquad \forall j \in J$$

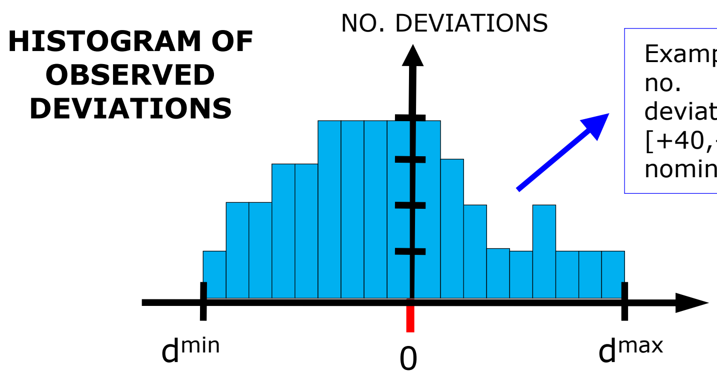
ROBUST COUNTERPART [Bertsimas, Sim 04] (LINEAR AND COMPACT)

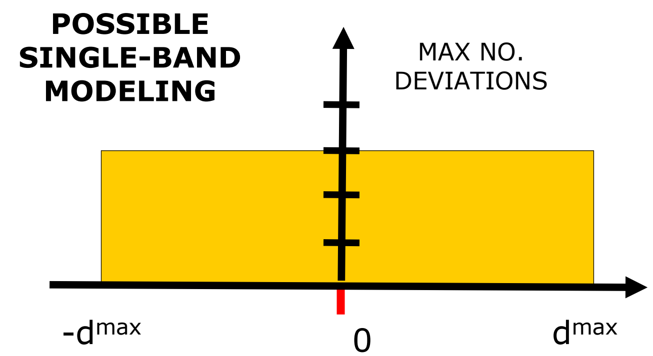
$$\max \sum_{j \in J} c_j x_j$$
$$\sum_{j \in J} \bar{a}_{ij} x_j \boxed{+ \Gamma_i w_i + \sum_{j \in J} z_{ij}} \le b_i \qquad \forall i \in I$$
$$w_i + z_{ij} \ge d_i^{\max} j x_j \qquad \forall i \in I, j \in J$$
$$z_{ij} \ge 0 \qquad \forall i \in I, j \in J$$
$$w_i \ge 0 \qquad \forall i \in I$$
$$x_j \ge 0 \qquad \forall j \in J$$

Using the BS model in practice

- In real-world problems, historical data about the deviations of the uncertain coefficients are commonly available
- The data can be easily used to build histograms representing the distribution of the deviations

? ARE WE REALLY ABLE TO EXPLOIT SUCH INFORMATION WITH THE BERTSIMAS-SIM MODEL ?





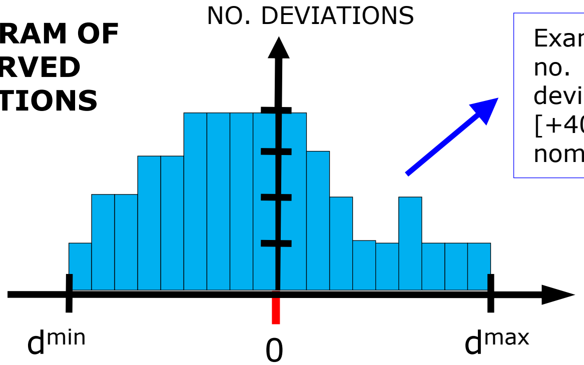
- The behaviour of the uncertainty internally to the deviation range is completely neglected (focus on the extreme deviations)
- According to our past experiences, practitioners would definitely prefer a more refined representation of the uncertainty

Multiband uncertainty (MB)

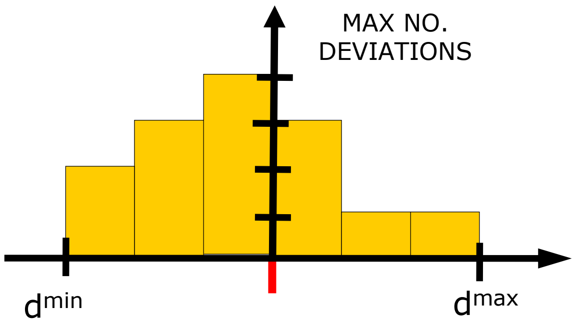
? WHAT CAN WE DO TO INCREASE OUR MODELING CAPACITY ?

ADOPT A MULTI-BAND UNCERTAINTY SET

HISTOGRAM OF OBSERVED DEVIATIONS



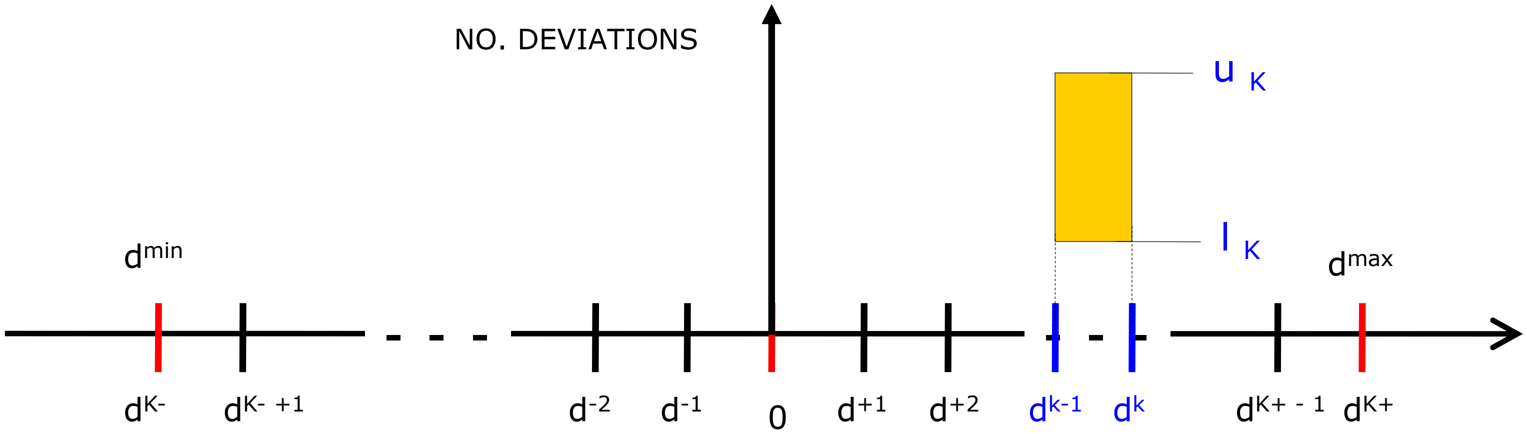
Example:
no. of coefficients deviating between [+40,+50]% from the nominal value



- strongly data-driven uncertainty set
- a general theoretical study was missing! → OUR AIM HAS BEEN TO FILL SUCH GAP

Formalizing Multiband Uncertainty

Focus on the coefficients a_{ij} of each constraint i (row-wise uncertainty)



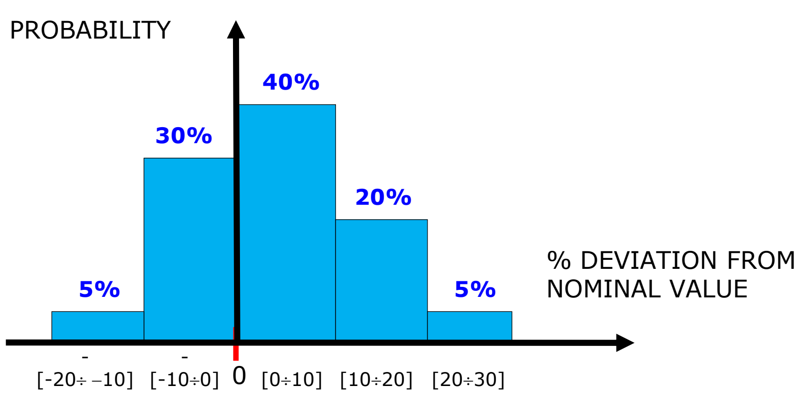
- K deviation values $-\infty < d_{ij}^{K-} < \cdots < d_{ij}^{0} = 0 < \cdots < d_{ij}^{K+} < +\infty$ for each coefficient a_{ij}
- K deviation bands such that each band k corresponds with range $(d_{ij}^{k-1}, d_{ij}^{k}]$
- Lower and upper bounds $0 \leq l_k \leq u_k \leq n$ on the number of coefficients deviating in each band k
- No upper bound on band k = 0, i.e. $u_0 = n$
- There exists a feasible assignment $\sum_{k \in K} l_k \leq n$

General example of construction

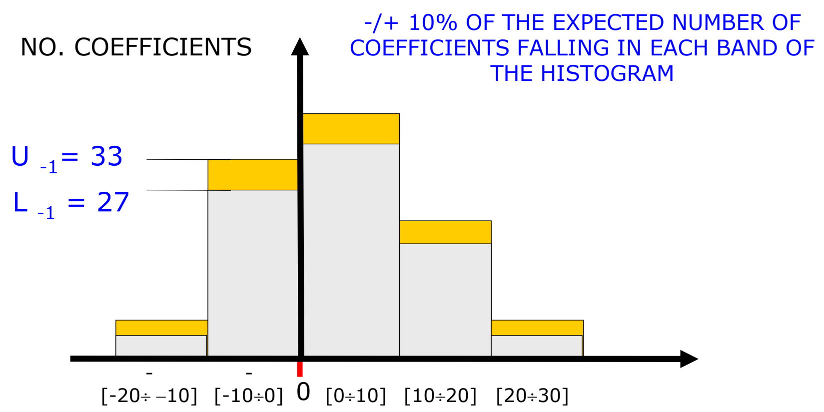
- Focus on the coefficients a_{ij} of each constraint i (row uncertainty)
- For each coefficient a_{ij}, we have a number of past observations $\hat{a}_{ij}$
- Compute the percentage deviation of an observation from the nominal value $\frac{\hat{a}_{ij} - \bar{a}_{ij}}{\bar{a}_{ij}} \cdot 100$
- Build the histogram representing the distribution of the percentage deviations for the considered constraint

Example

OBSERVED DISCRETE DISTRIBUTION (ALL COEFFICIENTS IN THE CONSTRAINT)



POSSIBLE MULTI-BAND SET FOR THE CONSTRAINT (assuming 100 coefficients in the constraint)



The max-deviation auxiliary problem under MB

MILP

NON-LINEAR ROBUST COUNTERPART

$$\max \sum_{j \in J} c_j x_j$$

$$\sum_{j \in J} \bar{a}_{ij} x_j \quad \leq b_i \qquad \forall i \in I$$

$$x_j \geq 0 \qquad \forall j \in J$$

$$x_j \in \mathbb{Z}_+ \qquad \forall j \in J_{\mathbb{Z}} \subseteq J$$

DEV01

$$\max \sum_{j \in J} \sum_{k \in K} d_{ij}^k \, x_j \, y_{ij}^k$$

MAXIMIZATION OF TOTAL DEVIATION

$$l_k \leq \sum_{j \in J} y_{ij}^k \leq u_k \qquad k \in K$$

BOUNDS ON THE NO. OF COEFFICIENTS FALLING IN BAND k

$$\sum_{k \in K} y_{ij}^k \leq 1 \qquad j \in J$$

EACH COEFFICIENT FALLS IN AT MOST ONE BAND

$$y_{ij}^k \in \{0, 1\} \qquad j \in J, k \in K$$

The Robust Counterpart under MB

PROPOSITION 1 (Büsing & D'Andreagiovanni 12)

The polytope associated with (DEV01) is integral.

- Proof based on showing that the coefficient matrix of (DEV01) is totally unimodular

THEOREM 1 (Büsing & D'Andreagiovanni 12)

The Robust Counterpart of (MILP) under multi-band uncertainty is equivalent to:

$$
\begin{array}{lll}
\max \sum_{j \in J} c_j \, x_j & & (RLP) \\
\sum_{j \in J} \bar{a}_{ij} \, x_j \boxed{- \sum_{k \in K} l_k \, v_i^k + \sum_{k \in K} u_k \, w_i^k + \sum_{j \in J} z_i^j} \leq b_i & i \in I & \\
-v_i^k + w_i^k + z_i^j \geq d_{ij}^k \, x_j & i \in I, j \in J, k \in K & \\
v_i^k, \; w_i^k \geq 0 & i \in I, k \in K & \\
z_i^j \geq 0 & i \in I, j \in J & \\
x_j \geq 0 & j \in J & \\
x_j \in \mathbb{Z}_+ & j \in J_{\mathbb{Z}} \subseteq J &
\end{array}
$$

- Proof based on exploiting the integrality of (DEV01) and strong duality
- If the original problem is linear, then also the counterpart is linear

Multiband Robustness by cutting planes

GOAL: finding a robust optimal solution for multi-band set D through a cutting-plane algorithm

Separation problem

Given a solution $x \in \mathbb{R}_+^{n-\lambda} \times \mathbb{Z}_+^{\lambda}$, is this solution robust feasible for constraint *i* ?

$x \in \mathbb{R}_+^{n-\lambda} \times \mathbb{Z}_+^{\lambda}$ robust feasible for i $\Longleftrightarrow$ $\sum_{j \in J} \bar{a}_{ij} x_j + DEV_i(x, D) \leq b_i$

If this condition does not hold and y* is an optimal solution to (DEV01) then

$$\sum_{j \in J} \bar{a}_{ij} x_j + \sum_{j \in J} \sum_{k \in K} d_{ij}^k \, x_j \, y_{ij}^{*k} \leq b_i$$

is a valid inequality for the original formulation and cuts off *x* (robustness cut)

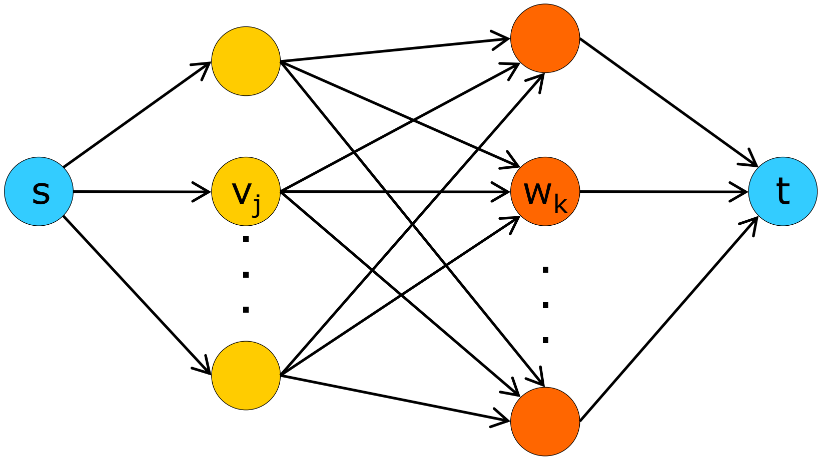
THEOREM 2 (Büsing & D'Andreagiovanni 12)

Separating a robustness cut corresponds with solving a min-cost flow problem

- Proof based on showing the 1:1 correspondence between integral flows and assignments y of (DEV01)

Efficient separation of robustness cuts

Solving (DEV01) is equivalent to solving a min-cost flow problem on the following graph (B. & D'A.12)



Send n unit of flows from s to t at minimum cost

SET OF NODES

- one node v_j for each variable x_j
- one node w_k for each band K
- source and sink

SET OF EDGES
(each associated with a triple (flow LB, flow UB, unitary cost))

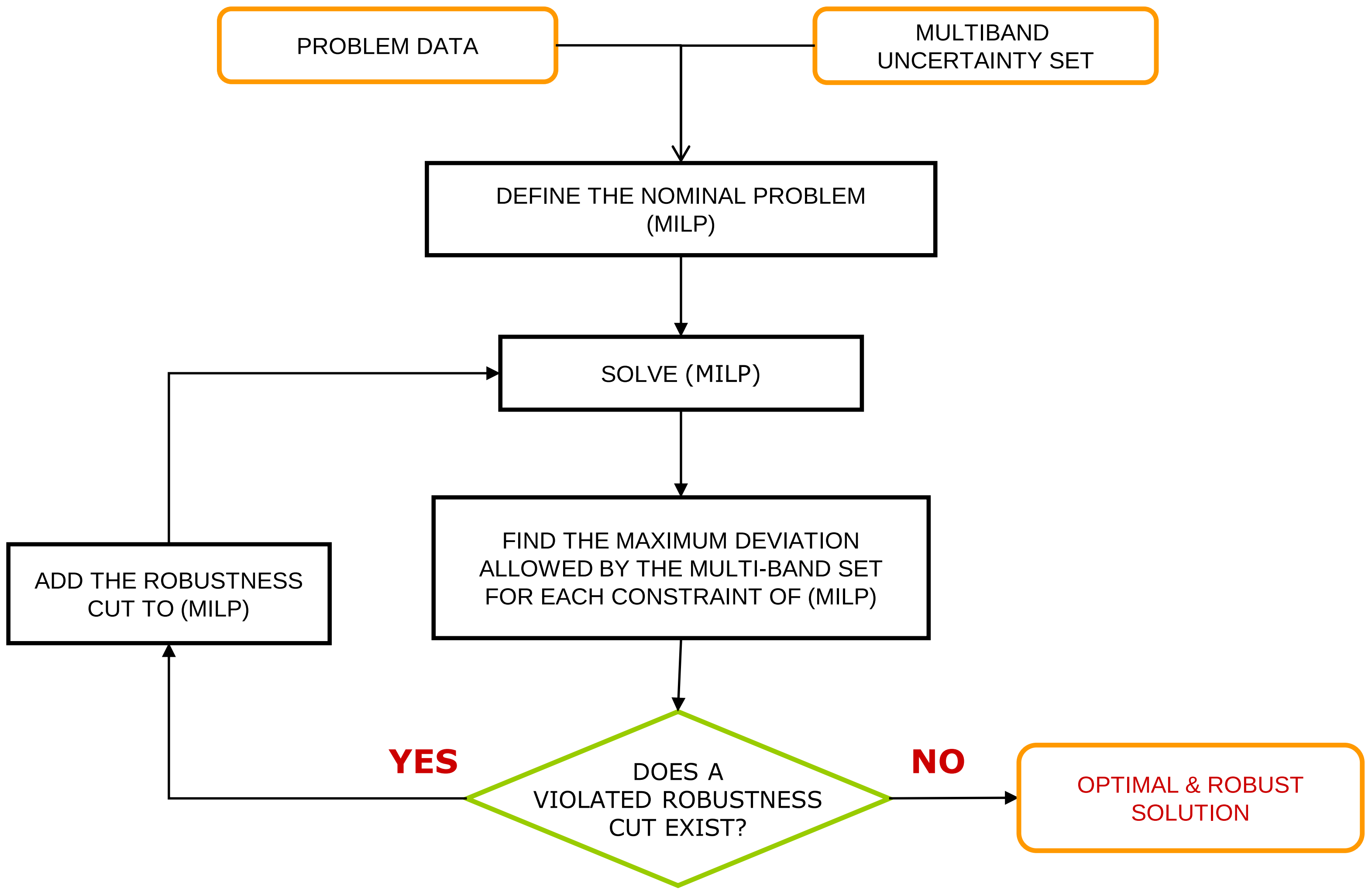
- one edge (s, v_j) with triple (0,1,0) for each x_j
- one edge (v_j, w_k) with triple $(0,1, - d_{ij}^k x_j)$ for each x_j and k
- one edge (w_k, t) with triple $(l_k, u_k, 0)$ for each k

PROPERTIES:

- 1:1 correspondence between integral flows of value n and complete assignments y of (DEV01)
- The cost relation between corresponding integral flow and assignment is: $d(x,y) = - c(f)$
- The following chain of equalities holds:

$$DEV_i(x, D) = \max_{y_i \in \bar{Y}_i} d(x, y_i) = - \min_{y_i \in \bar{Y}_i} -d(x, y_i) = - \min_{f \in F_i} c(f) = -c_i^*(x)$$

Basic robust cutting plane algorithm



0-1 Programs with Multiband cost uncertainty

$$\min\ c'\,x$$
$$x \in X \subseteq \{0,1\}^n$$

UNCERTAIN COST VECTOR

FEASIBLE SET = SUBSET OF ALL THE 0-1 VECTORS

$$\min \sum_{j\in J} c_j\,x_j + \sum_{k\in K} \theta_k\,w_k + \sum_{j\in J} z_j$$
$$w_k + z_j \geq d_j^k\,x_j \qquad j \in J, k \in K$$
$$w_k \geq 0 \qquad k \in K$$
$$z_j \geq 0 \qquad j \in J$$
$$x \in X,$$

THEOREM (Büsing & D'Andreagiovanni 12)

The robust optimal solution can be obtained by solving a polynomial number of nominal problems with modified cost coefficients. Tractability and approximability of the algorithm used to solve the nominal problem are preserved.

NOT JUST a trivial extension of the Bertsimas-Sim results and proofs!

Multiband Robustness - further results

- **Dominance among multiband uncertainty sets**
  (great reduction in the compact robust counterpart size)
  (Büsing & D’Andreagiovanni 2012)

- **Probability bounds** of constraint violation
  (Büsing & D’Andreagiovanni 2012)

- **(Strong) valid inequalities** for 0-1 Linear Programs
  (D‘Andreagiovanni & Raymond, 2013)

- **Robust cutting planes** for 0-1 linear programs with correlated uncertain right-hand-sides
  (D‘Andreagiovanni, 2014)

Comparing Gamma and Multiband Robustness

- Multiband linearity requires: (m n) **K** additional constraints
  (m + m n) **K** additional variables

- In the general case, MB can be less or more conservative than BS, depending upon the multiband structure (in our computational experience MB was always less conservative, when using realistic multiband sets and comparing them with realistic and fair Gamma-parameter)

- Anyway, we can derive some sufficient conditions for MB to be less conservative

  (by using majorizations/minorizations that **however reduces the actual advantage of MB over BS**)

$$\sum_{j=1}^{\Gamma}\left(1-\frac{k[i]}{\bar{K}}\right) - \sum_{j=\Gamma+1}^{\sum_{k\in K^+}\theta_k} \frac{k[i]}{\bar{K}} \geq 0$$

BAND IN WHICH THE i-TH LARGEST COEFFICIENT FALLS

NUM. POSITIVE BANDS

Remarks:

- The condition is independent from the solution x

- Useful condition to check that “rational” histogram representations of major distributions like the exponential and the normal ensures that MB is less conservative than BS

Multiband Robustness - applications

ROBUST POWER ASSIGNMENT
IN WIRELESS NETWORKS
(fading uncertainty)

ROBUST WIRELESS
NETWORK JAMMING
(jamming uncertainty)

ROBUST TIME-OFFSET SETTING
IN DVB-T NETWORKS
(time propagation uncertainty)

MULTIBAND ROBUST OPTIMIZATION

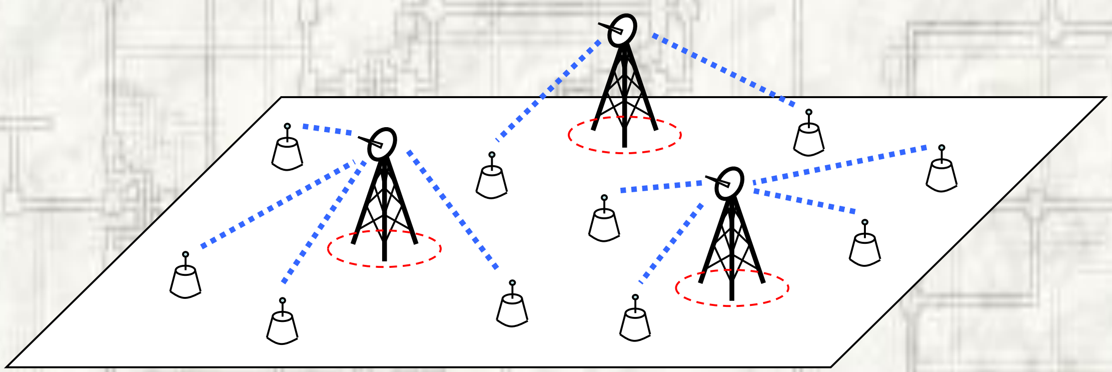
MULTIPERIOD CAPACITATED
NETWORK DESIGN
(traffic uncertainty)

ENERGY OFFERING FOR A PRICE-TAKER (price uncertainty)
Reduced price of robustness w.r..t. what you have seen at AIRO 2015

ROBUST HARVEST
SCHEDULING
(timber production uncertainty)

An application to Wireless Networks

A Wireless Network can be essentially described as a **set of transmitters T** which provide for a telecommunication service to a **set of receivers R** located in a target area



- Every transmitter is characterized by a **set of parameters**

Positional (antenna height, geographical location)

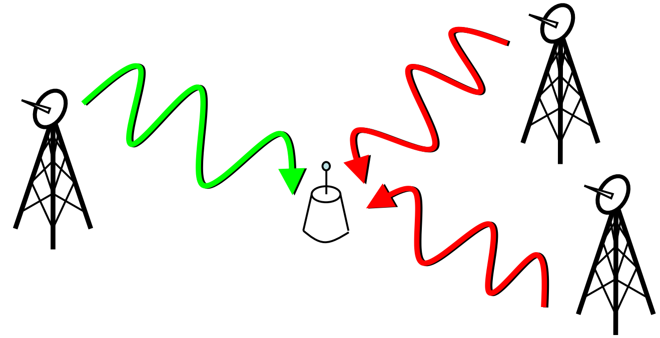
Radio-electrical (e.g., power emission, frequency channel)

WIRELESS NETWORK DESIGN PROBLEM (WND)

set the values of the parameters of each transmitter to maximize a profit function, while ensuring a minimum quality of service for each served receiver

Service coverage (1)

Every receiver r picks up signals from all the transmitters,



BUT:

- coverage is provided by a single transmitter, chosen as server of *r*
- all the other transmitters interfere the serving signal

If we introduce a continuous variable $0 \leq p_t \leq P^{\max}$ to represent power emission of transmitter t,

r is covered if the signal-to-interference ratio (SIR) is higher than a given threshold:

POWER RECEIVED FROM SERVER Tx

$$\frac{a_{r\sigma} \cdot p_\sigma}{\sum_{t \in T \setminus \{\sigma\}} a_{rt} \cdot p_t} \geq \delta$$

SUM OF POWER FROM INTERFERING Txs

COVERAGE THRESHOLD

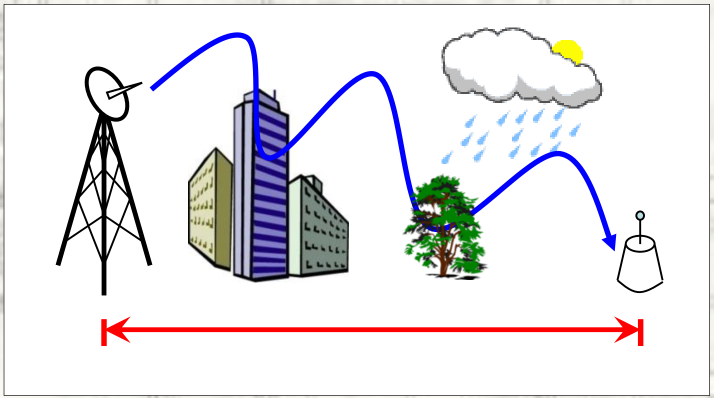
$$a_{r\sigma} \cdot p_\sigma - \delta \sum_{t \in T \setminus \{\sigma\}} a_{rt} \cdot p_t \geq 0$$

(SIR constraint)

Propagation and fading

A fading coefficient a_{rt} is usually computed through a propagation model and depends on several factors such as:

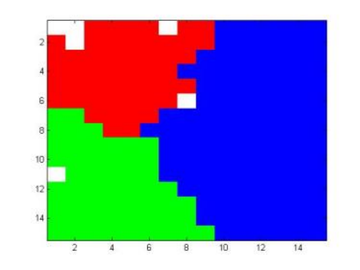
- the distance between t and r
- the presence of obstacles
- the weather

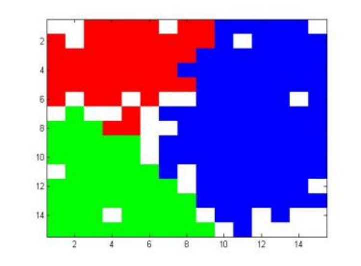


The fading coefficients are naturally subject to **uncertainty**

Neglecting uncertainty may lead to plans with **unexpected coverage holes**

EXPECTED COVERAGE





ACTUAL COVERAGE

Robust Power Assignment Problem

POWER ASSIGNMENT PROBLEM (PAP) → set the power emission of each transmitter to provide coverage to a set of receivers while minimizing the total power emission

$$\min \sum_{t \in T} p_t$$ **POWER MINIMIZATION**

$$a_{r\sigma} \cdot p_\sigma - \delta \sum_{t \in T \setminus \{\sigma\}} a_{rt} \cdot p_t \quad - \text{DEV(a,p)} \quad \geq 0 \qquad r \in R, \sigma \in T$$ **SIR CONSTRAINTS**

$$0 \leq p_t \leq P^{\max} \qquad t \in T$$ **POWER BOUNDS**

We take into account fading uncertainty
by subtracting the worst power deviation
in the l.h.s. of each SIR constraint

- To solve this robust problem we can adopt **multiband robustness** and either:
  - solve its linear and robust counterpart
  - find a robust optimal solution by the robust cutting-plane approach

Computational experience

TEST-BED: 15 WiMAX instances with up to 180 transmitters and 2118 testpoints defined in collaboration with wireless network professionals

BT

- fading coefficients assumed to be independent log-normal random variables (ITU Recommendation)
- 5 deviations bands (2 negative, 2 positive)
- all instances solved within one hour (Cplex 12.1, 4GB RAM)

INSTANCE ID

NO. CONSTRAINTS AND VARIABLES (nominal problem)

NO. CONSTRAINTS AND VARIABLES (compact robust counterpart)

PRICE OF ROBUSTNESS % (Gamma-Robustness)

REDUCTION IN THE PRICE OF ROBUSTNESS % (Multiband Robustness wrt Gamma-Robustness)

ID	$\|I\|$	$\|J\|$	$\|I^+\|$	$\|J^+\|$	PoR% (BS)	Δ PoR% (MB)
D1	90	148	2664	983	19.5	-13.2
D2	98	191	3744	1239	23.2	-11.2
D3	100	312	6240	1748	15.9	-7.5
D4	100	459	9180	2336	27.2	-9.3
D5	103	552	11371	2789	29.1	-13.5
D6	149	1055	31439	7032	18.4	-12.3
D7	157	1167	36643	8113	21.0	-15.4
D8	162	1224	39657	8741	17.6	-9.5
D9	169	1328	45089	9862	20.4	-11.1
D10	171	1405	48051	10465	23.1	-13.8
D11	174	1611	56062	12082	21.3	-12.7
D12	174	1725	60030	12876	24.6	-10.4
D13	176	1797	63254	13530	27.9	-8.9
D14	180	1882	67752	14450	22.0	-7.8
D15	180	2118	76248	16149	19.8	-7.6

Concluding remarks

- **World is stochastic** and most of real-world optimization problems involve **uncertain data**
- **Robust Optimization** is a modern and effective paradigm for dealing with data uncertainty
- We introduced **Multiband Robust Optimization** to generalize and refine the Bertsimas-Sim model

FUNDAMENTAL RESULTS:

- **compact robust counterpart** (purely linear if the nominal problem is purely Linear)

- **efficient separation of robustness cuts** by min-cost flow

- **experiments on real-world problems** indicate a **sensible reduction in the price of robustness**

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