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Modern Numerical Methods for Computational Electromagnetics

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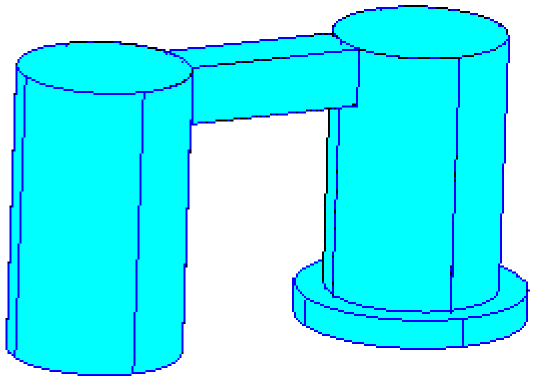
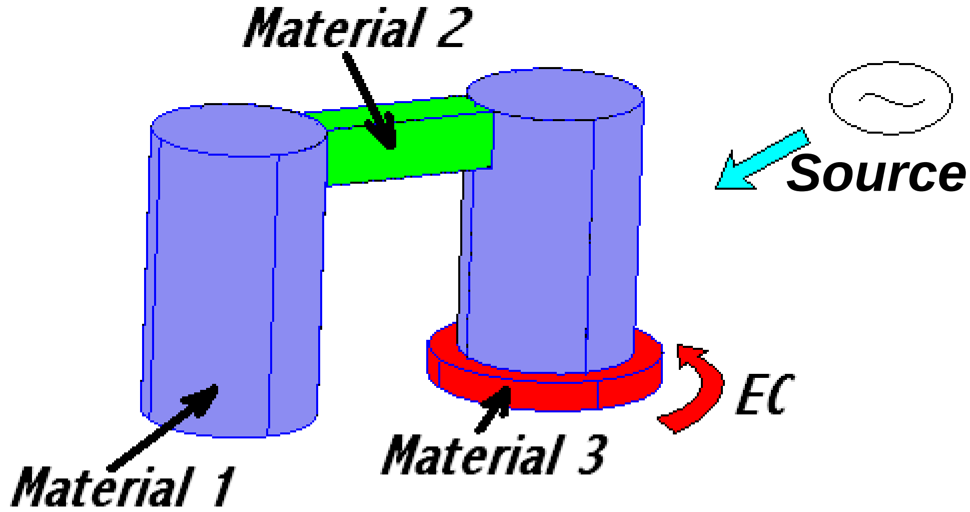
Outline

- Introduction
- Numerical methods for computing 3-D vector fields
  - Boundary value problem (BVP) – eddy-current analysis
  - Boundary value problem (BVP) – wave propagation analysis
  - Overview: FEM, BEM, MMP, FEM-MMP, DG-FEM
  - Discussion: Possibilities, advantages, drawbacks, problems, etc.
- Applications
  - Electromagnetic transients in high voltage equipment
  - Electromagnetic fields in transformers and machines
  - Microwave and optical devices
- Outlook





Mathematical model

Partial Differential Equation (PDE):

$$D\vec{f}\,(\vec{x}) = 0,\ \vec{x} \in \Omega$$

Boundary Conditions (BC):

$$B\vec{f}\,(\vec{x}) = 0,\ \vec{x} \in \partial\Omega$$

Model parameters

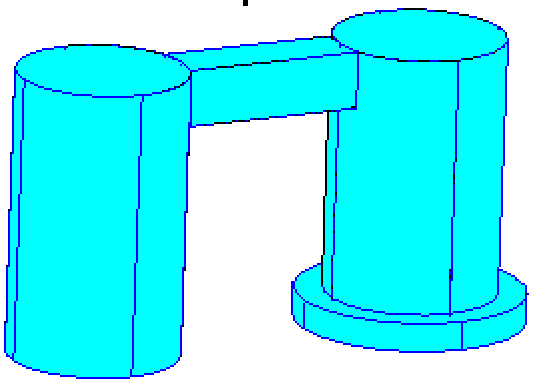
Material: $(\mu, \sigma, \varepsilon)$

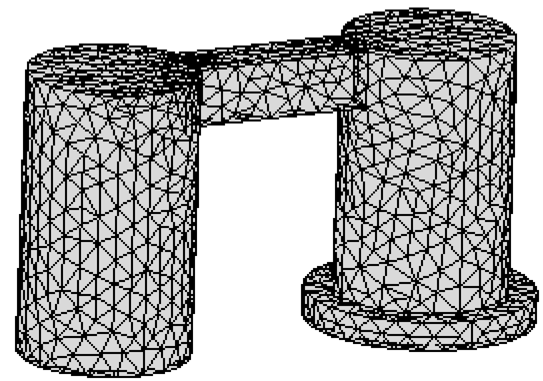
Frequency: (f)

Sources: $(\vec{E}_s, \vec{H}_s, \vec{J}_s)$

Introduction

Model discretization





PDE & Boundary Condition

$$D\vec{f}(\vec{x}) = 0,\ \vec{x} \in \Omega$$

$$B\vec{f}(\vec{x}) = 0,\ \vec{x} \in \partial\Omega$$

Numerical method

Large linear system of equations

$$[A]\{x\} = \{b\}$$

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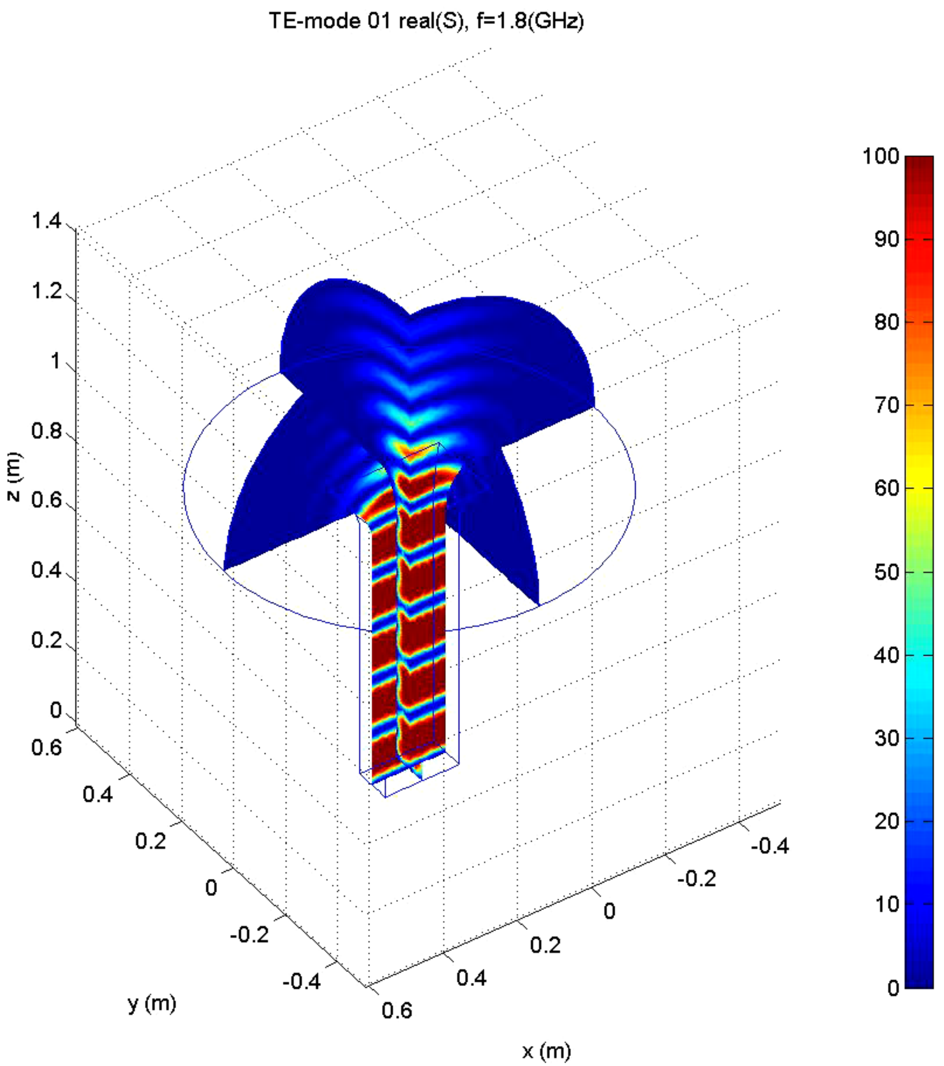

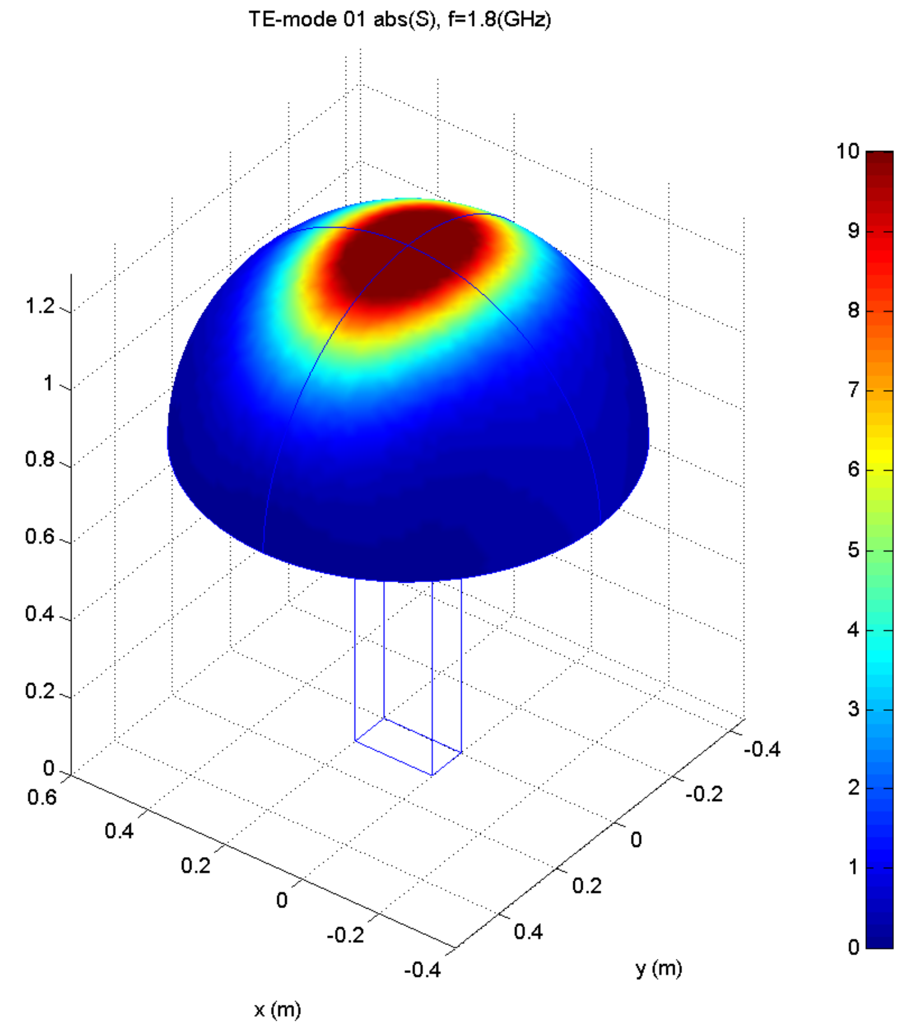

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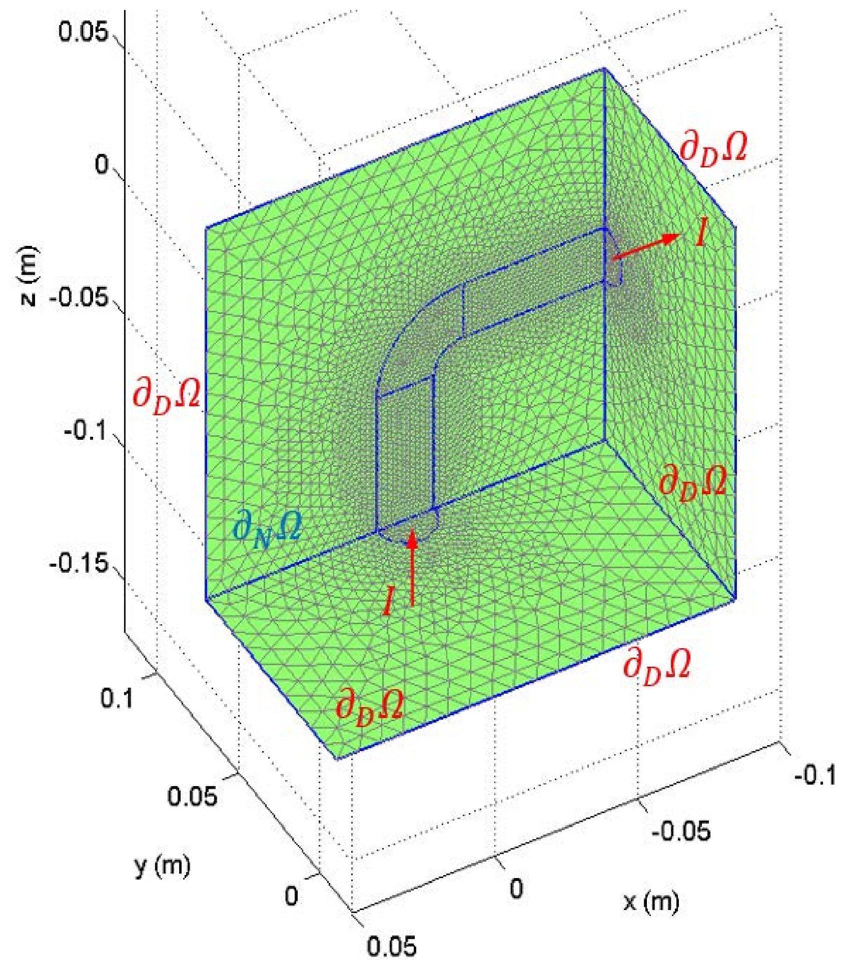
BVP for Eddy-current Analysis[1]

$$\nabla\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right)+j\omega\sigma\vec{A}=\vec{J}_S,\ in\,\Omega \qquad (1)$$

$$\vec{n}\times\vec{A}=0,\ over\ \partial_D\Omega \qquad (2)$$

$$\vec{n}\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right)=0,\ over\ \partial_N\Omega \qquad (3)$$



[1] J. Smajic, "How to Perform Electromagnetic Finite Element Analysis", the International Association for the Engineering Modelling, Analysis & Simulation Community, NAFEMS Ltd., Hamilton, UK, 2016.

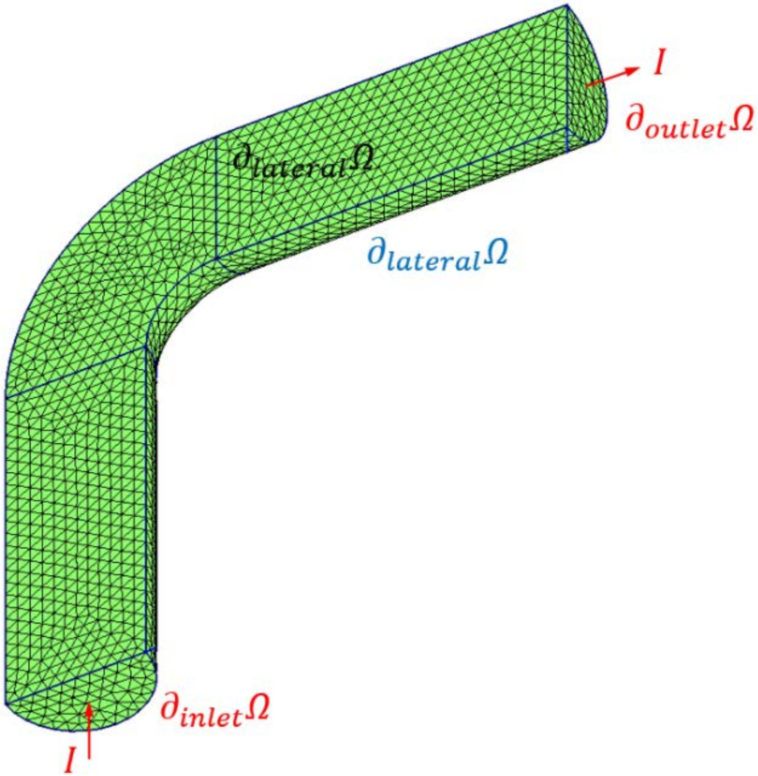
BVP for Stationary Current Distribution[1]

$$\nabla\left(\sigma \nabla \varphi\right) = 0,\ in\ \Omega_c \tag{22}$$

$$\varphi = 0,\ over\ \partial_{outlet}\Omega_c \tag{23}$$

$$\sigma \frac{\partial \varphi}{\partial n} = -\frac{I}{S_{inlet}},\ over\ \partial_{inlet}\Omega_c \tag{24}$$

$$\sigma \frac{\partial \varphi}{\partial n} = 0,\ over\ \partial_{lateral}\Omega_c \tag{25}$$



[1]J. Smajic, "How to Perform Electromagnetic Finite Element Analysis", the International Association for the Engineering Modelling, Analysis & Simulation Community, NAFEMS Ltd., Hamilton, UK, 2016.

BVP for Wave Propagation Analysis[1]

$$\nabla\times\left(\frac{1}{\mu_r}\nabla\times\vec{E}\right)+j\omega\mu_0\sigma\vec{E}-\omega^2\mu_0\varepsilon\vec{E}=0 \text{ in } \Omega\subseteq R^3 \quad (1)$$

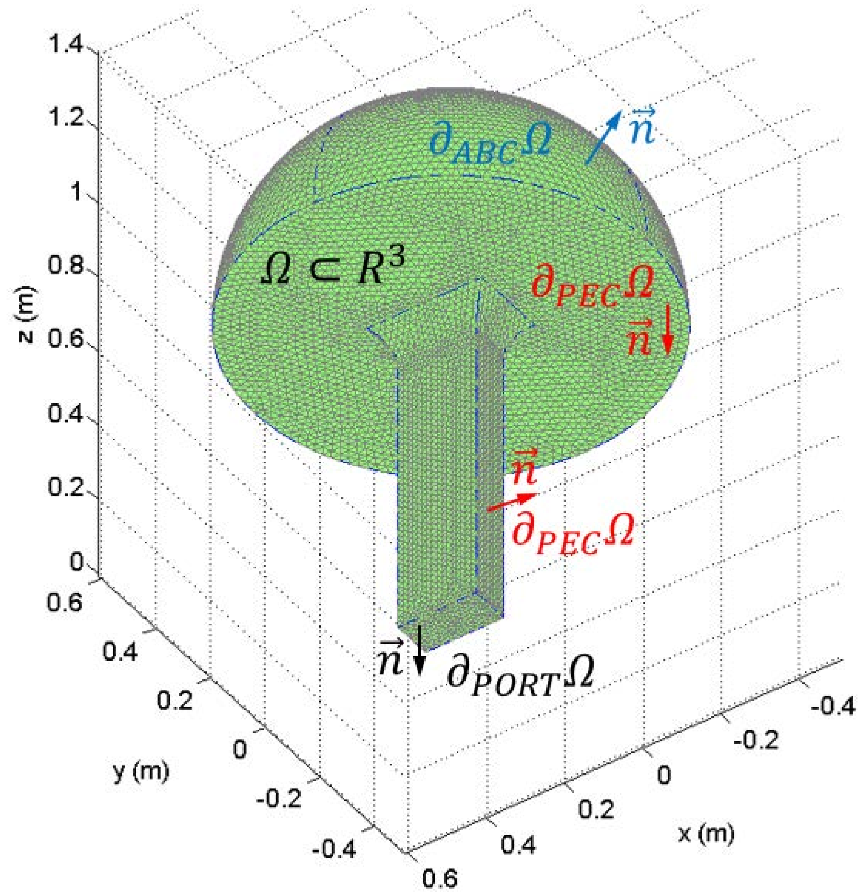
$$\vec{n}\times\vec{E}=0\text{, over }\partial_{PEC}\Omega \quad (2)$$

$$\begin{aligned}&\vec{n}_1\times\left(\nabla\times\vec{E}\right)+jk_z\vec{n}_1\times\left(\vec{n}_1\times\vec{E}\right)=\\&+2jk_z\,\vec{n}_1\times\left(\vec{n}_1\times\vec{E}_i\right)\end{aligned}\text{, over }\partial_{PORT1}\Omega \quad (3)$$

$$\vec{n}_2\times\left(\nabla\times\vec{E}\right)+jk_0\vec{n}_2\times\left(\vec{n}_2\times\vec{E}\right)=0\text{, over }\partial_{PORT2}\Omega \quad (4)$$

$$k_z=\sqrt{\omega^2\mu\varepsilon-k_t^2} \quad (6)$$

$$k_{tmn}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \quad (7)$$



[1]J. Smajic, "How to Perform Electromagnetic Finite Element Analysis", the International Association for the Engineering Modelling, Analysis & Simulation Community, NAFEMS Ltd., Hamilton, UK, 2016.

Vector Finite Element Method (FEM)

$$\iiint\limits_{(\Omega_c)} \vec{N}_i \cdot \nabla \times \left(\frac{1}{\mu_c} \nabla \times \vec{A}_c \right) dV + \iiint\limits_{(\Omega_c)} j\omega\sigma_c \vec{N}_i \cdot \vec{A}_c \, dV = \iiint\limits_{(\Omega_c)} \vec{N}_i \cdot \vec{J}_S \, dV \tag{4}$$

$$\oint\!\!\!\oint\limits_{(\partial\Omega_c)} \left[\vec{n}_c \times \left(\frac{1}{\mu_c} \nabla \times \vec{A}_c \right) \right] \cdot \vec{N}_i \, dS + \iiint\limits_{(\Omega_c)} \left(\frac{1}{\mu_c} \nabla \times \vec{A}_c \right) \cdot \left(\nabla \times \vec{N}_i \right) dV + \iiint\limits_{(\Omega_c)} j\omega\sigma_c \vec{N}_i \cdot \vec{A}_c \, dV = \iiint\limits_{(\Omega_c)} \vec{N}_i \cdot \vec{J}_S \, dV \tag{7}$$

$$\oint\!\!\!\oint\limits_{(\partial\Omega_a)} \left[\vec{n}_a \times \left(\frac{1}{\mu_a} \nabla \times \vec{A}_a \right) \right] \cdot \vec{N}_i \, dS + \iiint\limits_{(\Omega_a)} \left(\frac{1}{\mu_a} \nabla \times \vec{A}_a \right) \cdot \left(\nabla \times \vec{N}_i \right) dV = 0 \tag{10}$$

$$\vec{n}_c \cdot \vec{B}_c = \vec{n}_c \cdot \vec{B}_a \quad \Rightarrow \quad \vec{n}_c \cdot \left(\nabla \times \vec{A}_c \right) = \vec{n}_c \cdot \left(\nabla \times \vec{A}_a \right) \tag{12}$$

$$\vec{n}_c \times \vec{H}_c = \vec{n}_c \times \vec{H}_a \quad \Rightarrow \quad \vec{n}_c \times \left(\frac{1}{\mu_c} \nabla \times \vec{A}_c \right) = \vec{n}_c \times \left(\frac{1}{\mu_a} \nabla \times \vec{A}_a \right) \tag{13}$$

Vector Finite Element Method (FEM)

$$\vec{A}(x,y,z)=\sum_{j}\vec{N}_j(x,y,z)A_j \tag{18}$$

$$\Omega=\bigcup_{e=1}^{N_e}\Omega^e \tag{19}$$

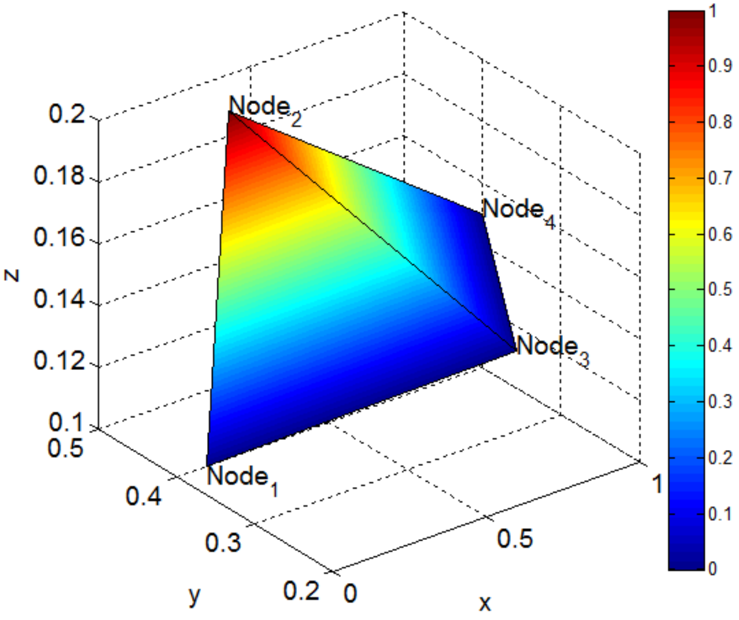
$$\sum_{j=1}^{N_{ed}}A_j\left[\sum_{\substack{e\\ i,j\in\Omega^e}}\iiint_{(\Omega^e)}\frac{1}{\mu_r^e}\left(\nabla\times\vec{N}_i\right)\cdot\left(\nabla\times\vec{N}_j\right)dV+j\omega\mu_0\sum_{\substack{e\\ i,j\in\Omega^e}}\iiint_{(\Omega^e)}\sigma^e\vec{N}_i\cdot\vec{N}_j\,dV\right]=\mu_0\sum_{\substack{e\\ i\in\Omega^e}}\iiint_{(\Omega^e)}\vec{N}_i\cdot\vec{J}_S^e\,dV \tag{20}$$

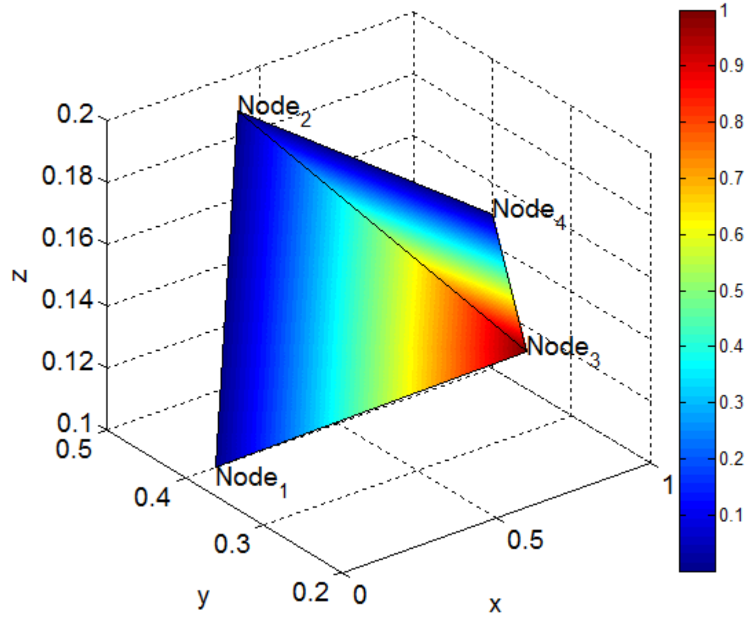
$$[K]\{A\}=\left([B]_{curl-curl}+[C]_\sigma\right)\{A\}=\{b\} \tag{21}$$

Vector Finite Element Method (FEM)

$$\vec{A}(x,y,z)=\sum_j \vec{N}_j(x,y,z)A_j \tag{18}$$

$$\vec{N}_i^e(x,y)=l_i^e\cdot\left[N_{in_1}^e(x,y)\cdot\nabla N_{in_2}^e(x,y)-N_{in_2}^e(x,y)\cdot\nabla N_{in_1}^e(x,y)\right]$$





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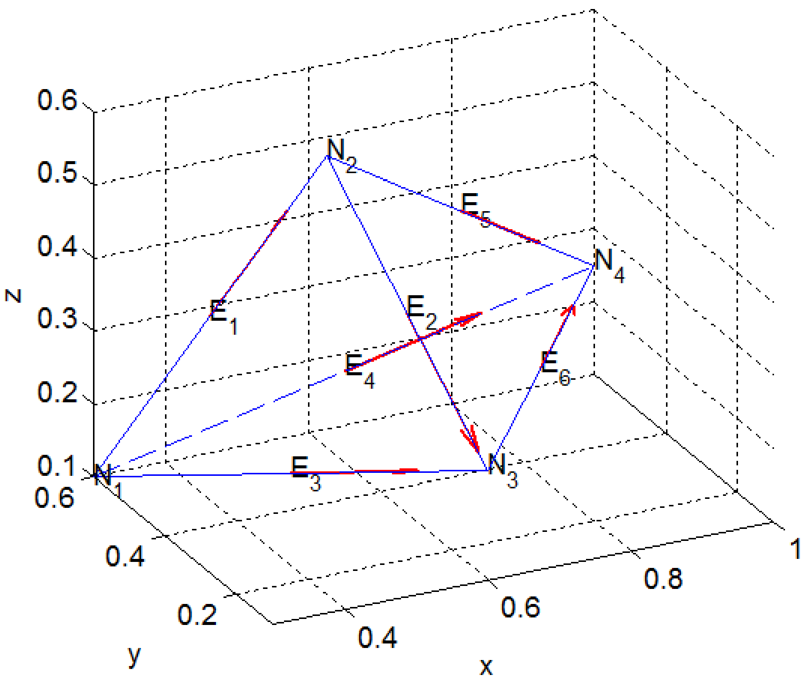
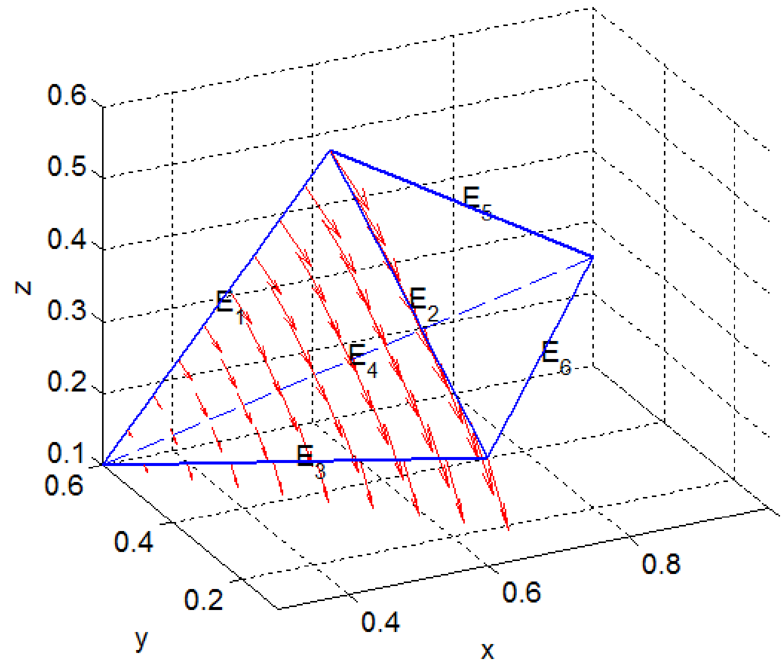
Vector Finite Element Method (FEM) [2,3]

$$\vec{A}(x,y,z)=\sum_{j}\vec{N}_j(x,y,z)A_j \qquad (18)$$

$$\vec{N}_i^e(x,y)=l_i^e\cdot\left[N_{in_1}^e(x,y)\cdot\nabla N_{in_2}^e(x,y)-N_{in_2}^e(x,y)\cdot\nabla N_{in_1}^e(x,y)\right]$$



Vector Shape Function: vN_2



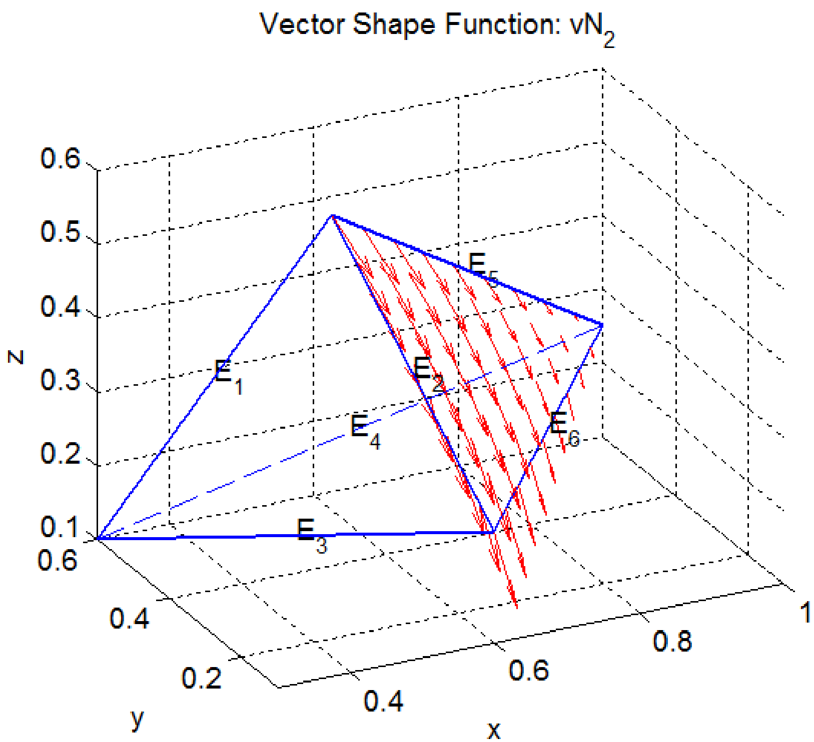
[2]H. Whitney, “Geometric integration theory”, Princeton University Press, Princeton, NJ, 1957.

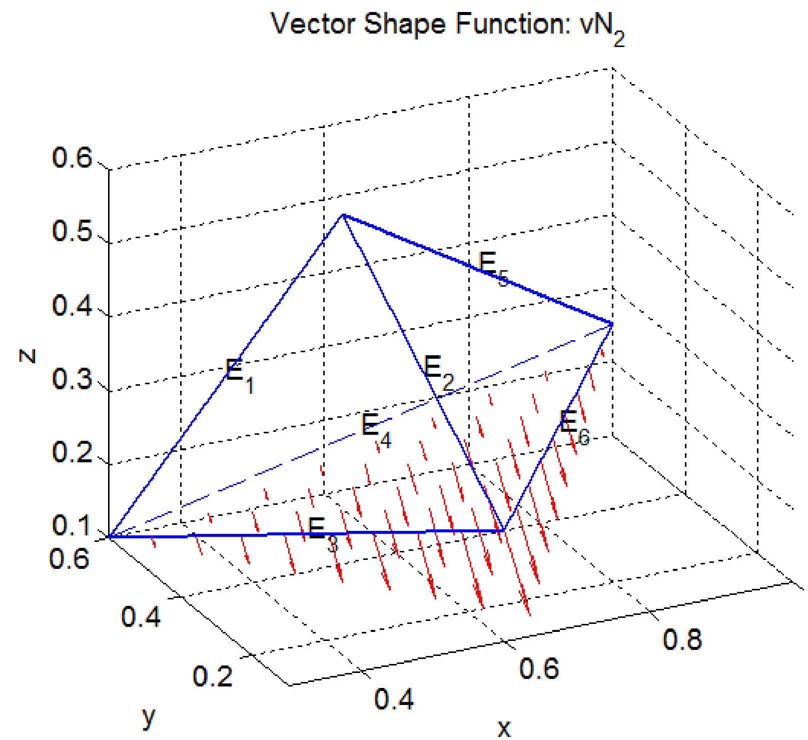
[3]J. C. Nedelec, “Mixed Finite Elements in R3”, Numer. Meth., Vol. 35, pp. 315-341, 1980.

Vector Finite Element Method (FEM)

$$\vec{A}(x,y,z)=\sum_j \vec{N}_j(x,y,z)\,A_j \tag{18}$$

$$\vec{N}_i^e(x,y)=l_i^e\cdot\left[N_{in_1}^e(x,y)\cdot\nabla N_{in_2}^e(x,y)-N_{in_2}^e(x,y)\cdot\nabla N_{in_1}^e(x,y)\right]$$

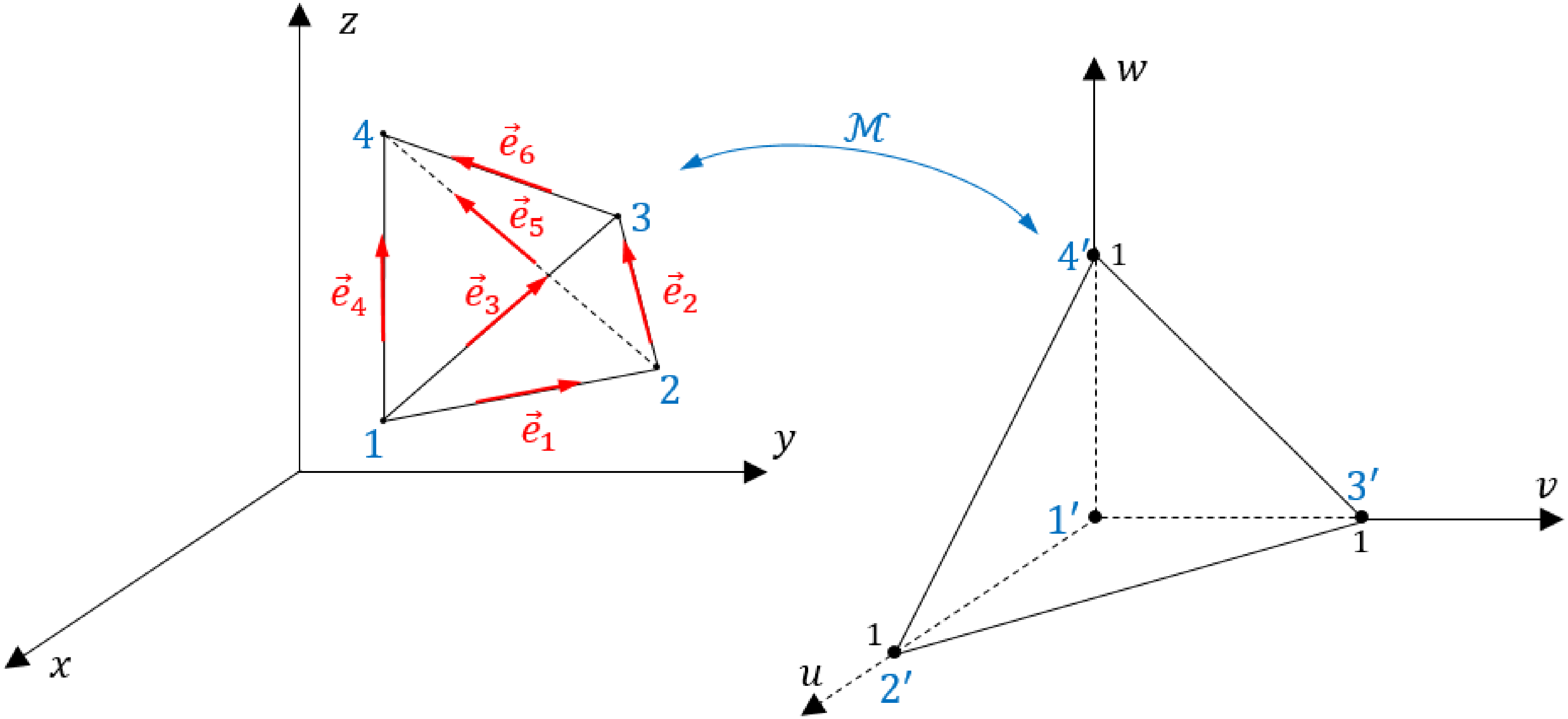




[2]H. Whitney, “Geometric integration theory”, Princeton University Press, Princeton, NJ, 1957.

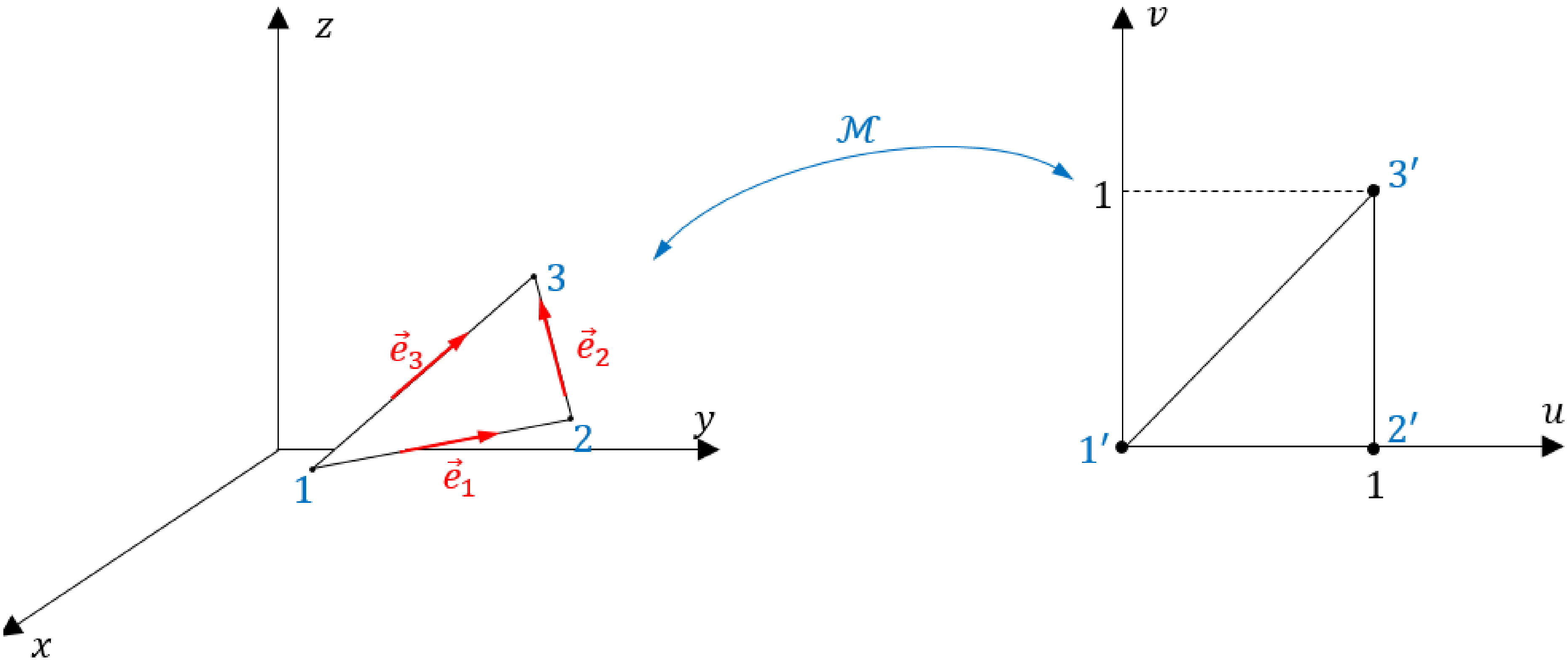
[3]J. C. Nedelec, “Mixed Finite Elements in R3”, Numer. Meth., Vol. 35, pp. 315-341, 1980.

Numerical volume integration



$$\iiint\limits_{(V_e)} f(x,y,z)dV = \int_0^1\int_0^u\int_0^u f(u,v,w)|J|du\,dv\,dw = \int_0^1\int_0^u\int_0^u f(u,v,w)\frac{V_e}{6}du\,dv\,dw$$

Numerical surface integration



$$\iint_{(\Delta^e)} f(x,y,z)dS = \int_0^1 \int_0^u f(u,v)|J|\,du\,dv = \int_0^1 \int_0^u f(u,v)2\Delta_e\,du\,dv$$

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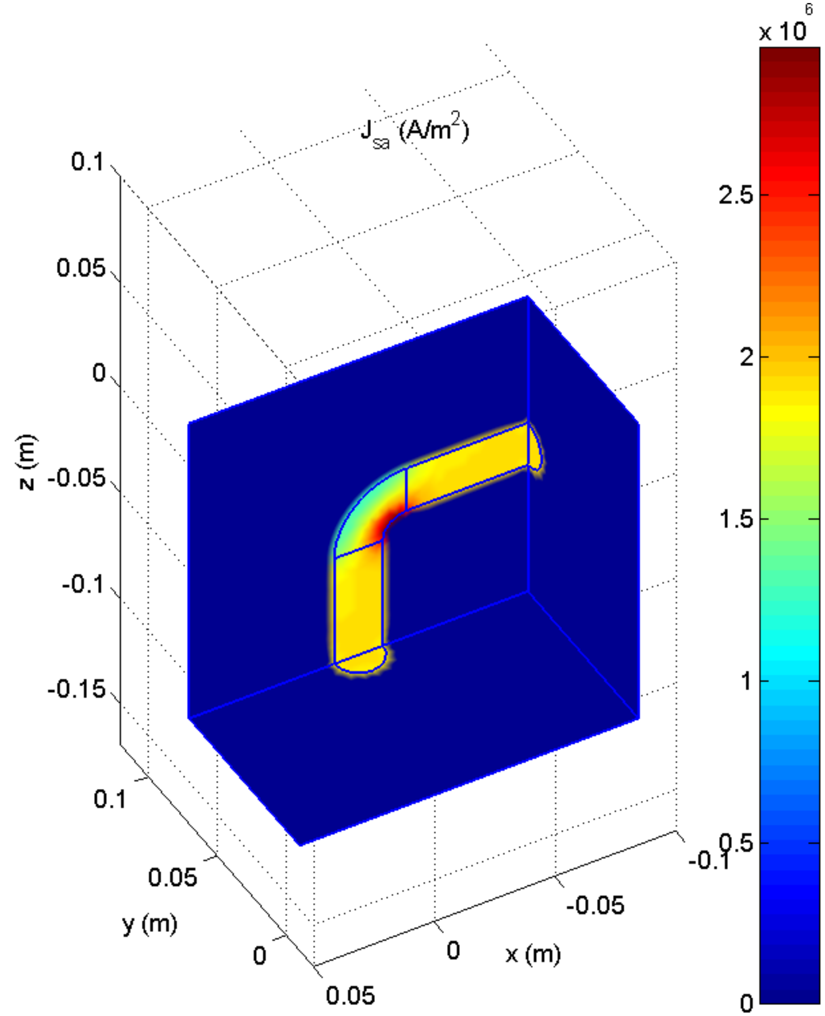
Vector Finite Element Method (FEM)

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) + j\omega\sigma\vec{A} = \vec{J}_S, \ in\, \Omega \qquad (1)$$

$$\vec{n} \times \vec{A} = 0, \ over\ \partial_D \Omega \qquad (2)$$

$$\vec{n} \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = 0, \ over\ \partial_N \Omega \qquad (3)$$



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Vector Finite Element Method (FEM)

$$\sigma_{air} = 0(S/m), \sigma_{cu} = 3.5 \cdot 10^7 (S/m)$$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) + j\omega\sigma\vec{A} = \vec{J}_S, \; in\, \Omega \qquad (1)$$

$$\vec{n} \times \vec{A} = 0, \; over\; \partial_D \Omega \qquad (2)$$

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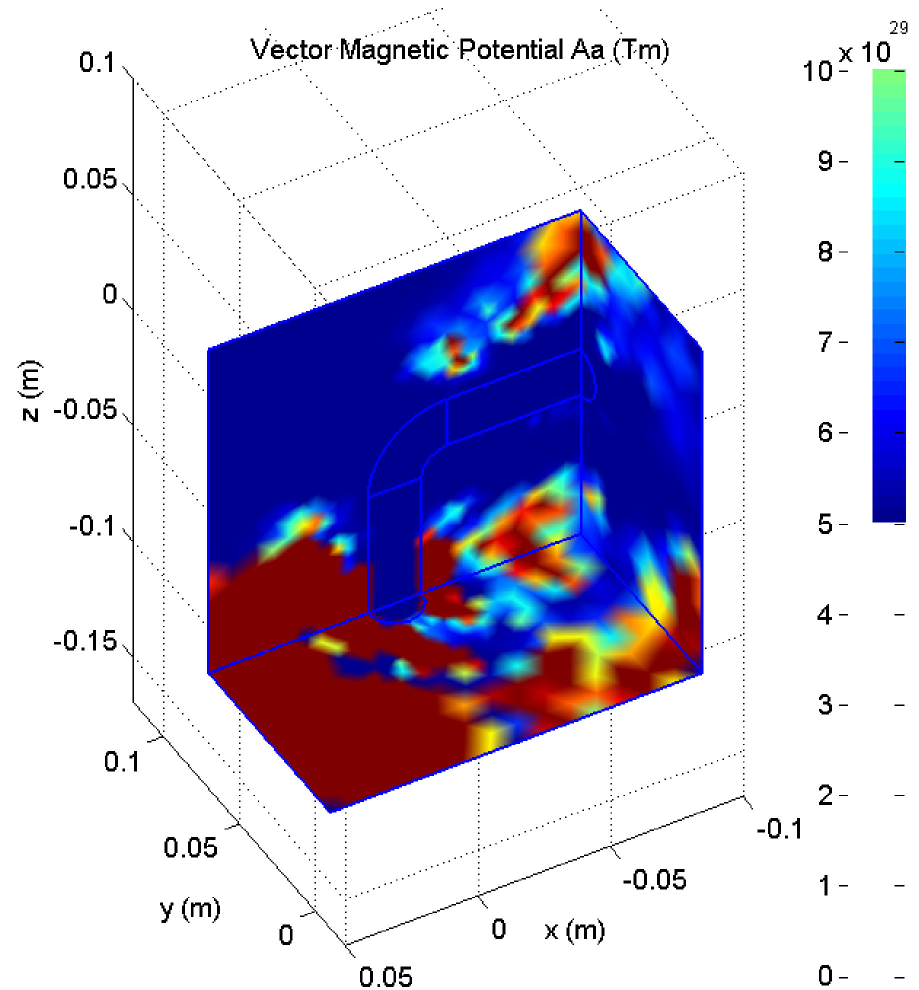

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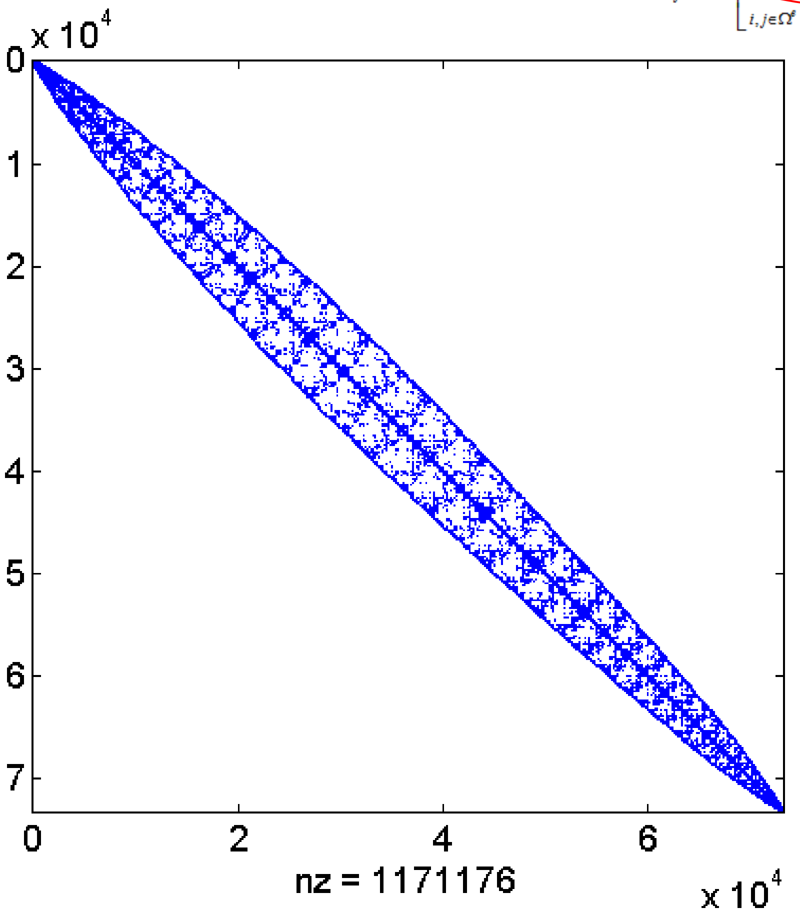
Vector Finite Element Method (FEM)

$$\sum_{j=1}^{N_{ed}} A_j \left[\overbrace{\sum_{\substack{e \\ i,j\in\Omega^e}} \iiint_{(\Omega^e)} \frac{1}{\mu_r^e} \left(\nabla\times\vec{N}_i\right)\cdot\left(\nabla\times\vec{N}_j\right) dV}^{\boldsymbol{B_{ij}}} + \overbrace{j\omega\mu_0 \sum_{\substack{e \\ i,j\in\Omega^e}} \iiint_{(\Omega^e)} \sigma^e \vec{N}_i\cdot\vec{N}_j \, dV}^{\boldsymbol{C_{ij}}} \right] = \mu_0 \sum_{\substack{e \\ i\in\Omega^e}} \iiint_{(\Omega^e)} \vec{N}_i\cdot\vec{J}_S^e \, dV \qquad (20)$$



N_{edges}=73’276

$$Density = \frac{N_{nz}}{N_{edges}^2}\cdot 100\% = 0.022\%$$

$$\sigma_{air} = 10^4 (S/m), \sigma_{cu} = 3.5\cdot 10^7 (S/m)$$

$$\kappa(B) = \|B\|\cdot\|B^{-1}\| = 8.04\cdot 10^{26}$$

$$\kappa(C) = \|C\|\cdot\|C^{-1}\| = 1.91\cdot 10^{5}$$

$$\kappa(K) = \|K\|\cdot\|K^{-1}\| = \mathbf{3.38\cdot 10^{25}}$$

Vector Finite Element Method (FEM)

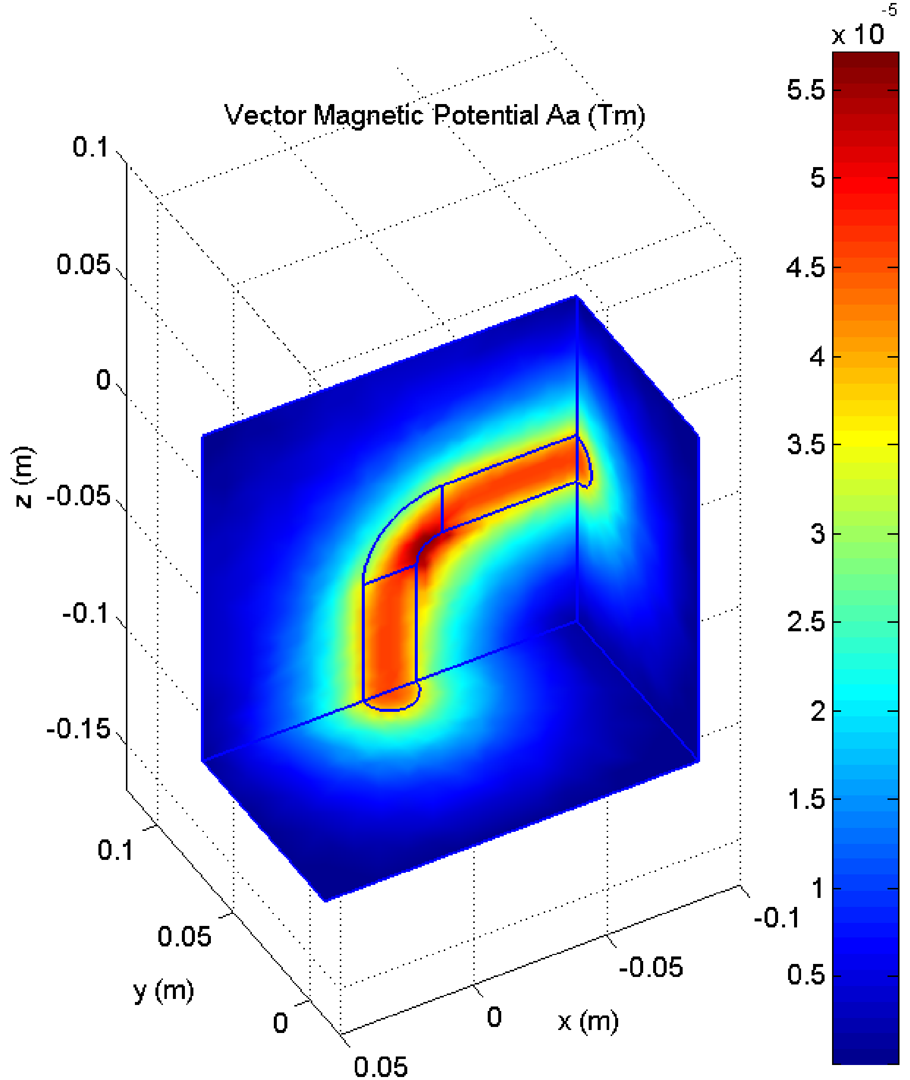
Unrealistic improvement:

$\sigma_{air} = 10^4 (S/m), \sigma_{cu} = 3.5 \cdot 10^7 (S/m)$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) + j\omega\sigma\vec{A} = \vec{J}_S, \; in\, \Omega \qquad (1)$$

$$\vec{n} \times \vec{A} = 0, \; over\; \partial_D \Omega \qquad (2)$$

$$\vec{n} \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = 0, \; over\; \partial_N \Omega \qquad (3)$$



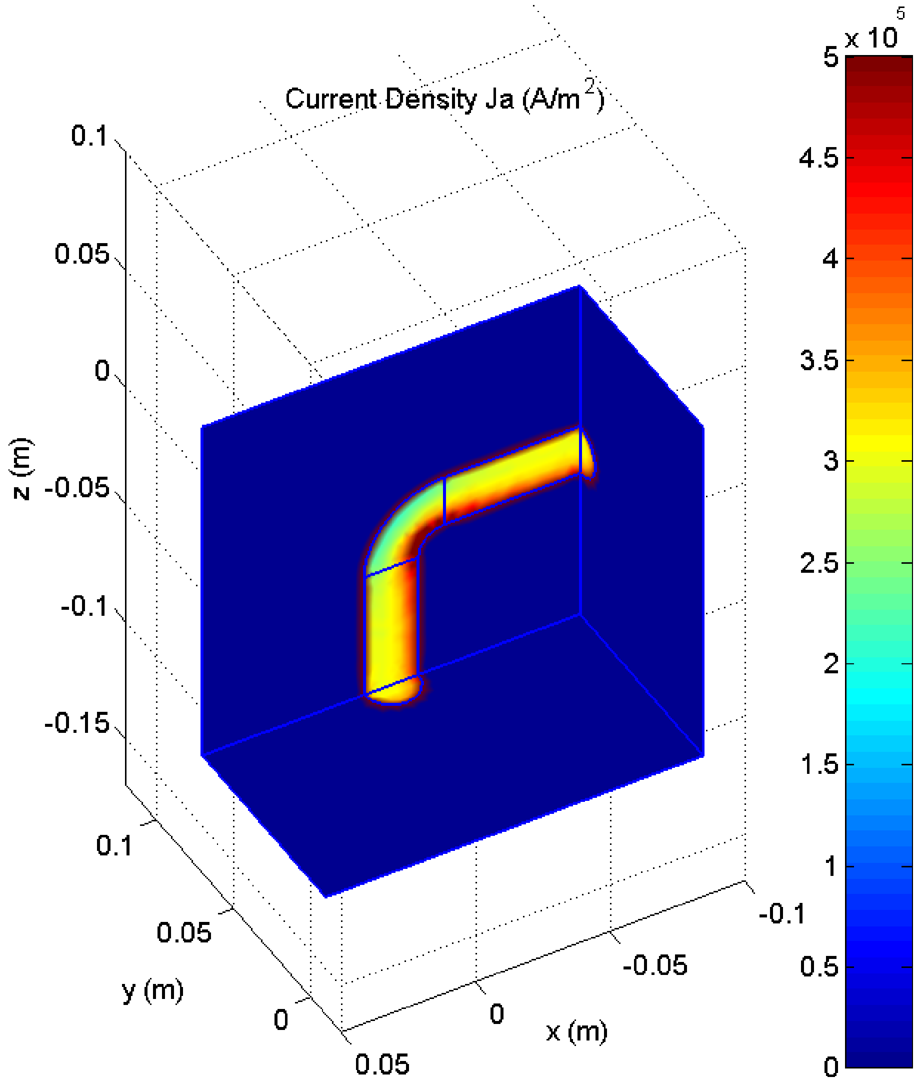
Vector Finite Element Method (FEM)

$$\nabla\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right)+j\omega\sigma\vec{A}=\vec{J}_S,\ in\,\Omega \qquad (1)$$

$$\vec{n}\times\vec{A}=0,\ over\ \partial_D\Omega \qquad (2)$$

$$\vec{n}\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right)=0,\ over\ \partial_N\Omega \qquad (3)$$

$$\vec{J}=\vec{J}_S+\vec{J}_{EC}=\vec{J}_S+\sigma\vec{E}_{EC}=\vec{J}_S-j\omega\sigma\vec{A}$$



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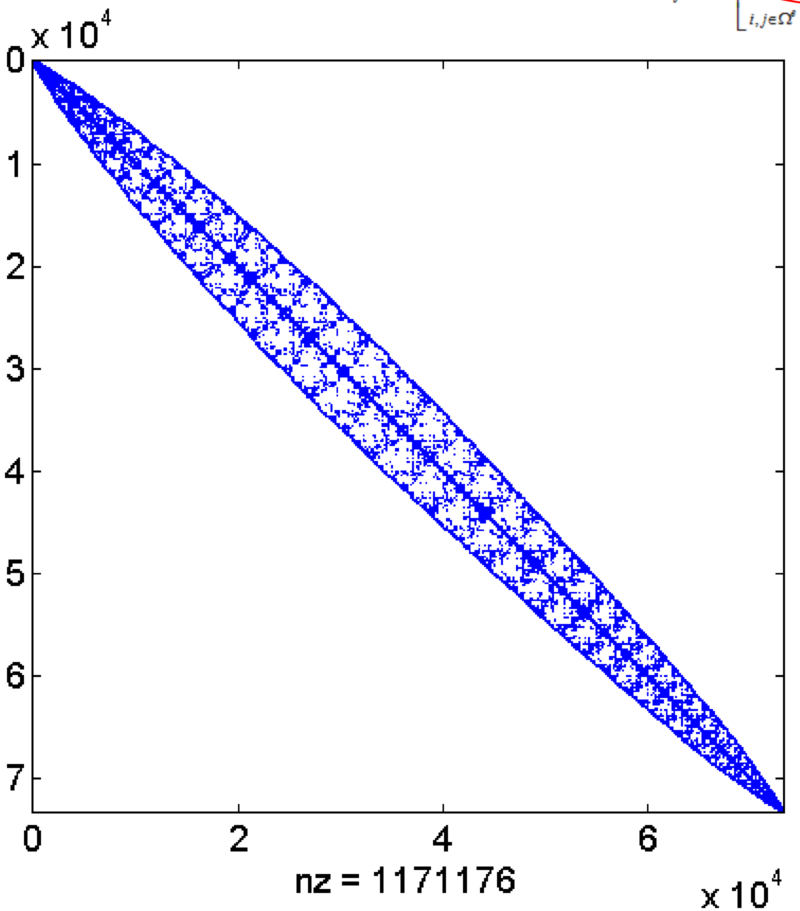
Vector Finite Element Method (FEM)

$$\sum_{j=1}^{N_{ed}} A_j \left[\overbrace{\sum_{\substack{e\\ i,j\in\Omega^e}} \iiint_{(\Omega^e)} \frac{1}{\mu_r^e}\left(\nabla\times\vec{N}_i\right)\cdot\left(\nabla\times\vec{N}_j\right) dV}^{B_{ij}} + \overbrace{j\omega\mu_0 \sum_{\substack{e\\ i,j\in\Omega^e}} \iiint_{(\Omega^e)} \sigma^e \vec{N}_i\cdot\vec{N}_j\, dV}^{C_{ij}} \right] = \mu_0 \sum_{\substack{e\\ i\in\Omega^e}} \iiint_{(\Omega^e)} \vec{N}_i\cdot\vec{J}_S^e\, dV \qquad (20)$$



N_{edges}=73’276

$$Density = \frac{N_{nz}}{N_{edges}^2}\cdot 100\% = 0.022\%$$

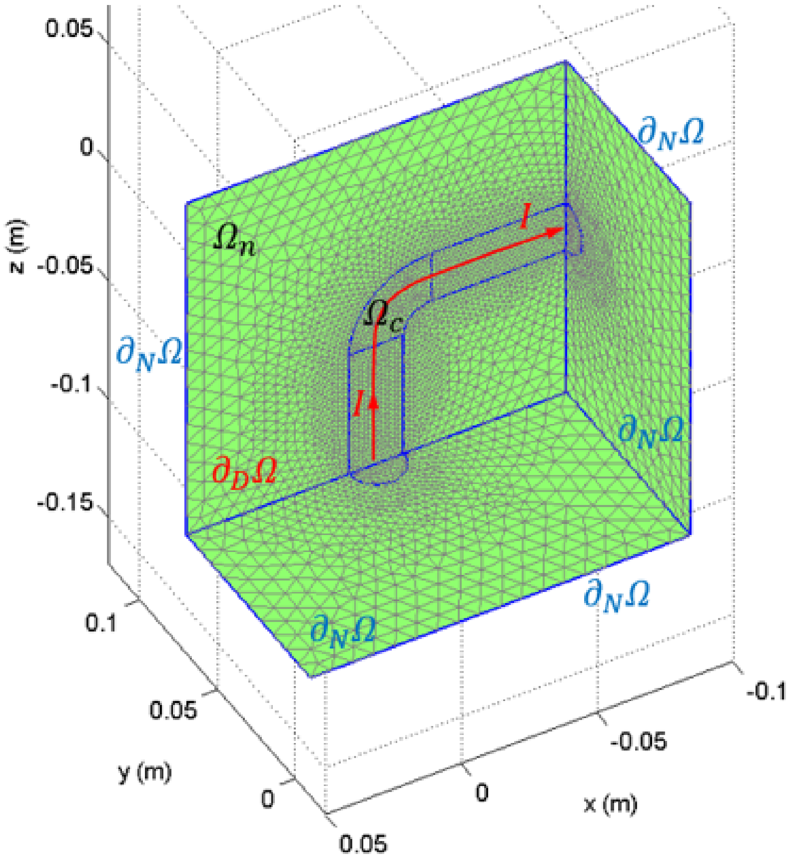
$$\sigma_{air} = 10^4 (S/m), \sigma_{cu} = 3.5\cdot 10^7 (S/m)$$

$$\kappa(B) = \|B\|\cdot\|B^{-1}\| = 8.04\cdot 10^{26}$$

$$\kappa(C) = \|C\|\cdot\|C^{-1}\| = 1.91\cdot 10^5$$

$$\kappa(K) = \|K\|\cdot\|K^{-1}\| = 2.88\cdot 10^7$$

Scalar + Vector Finite Element Method (FEM), $\vec{H} - \Phi$ field formulation



Conductor region

$$\frac{1}{\sigma}\nabla\times\vec{H}_c = \vec{E}_c,\ in\ \Omega_c \qquad (1)$$

$$\vec{n}\times\left(\frac{1}{\sigma}\nabla\times\vec{H}_c\right) = 0,\ over\ \partial_N\Omega_c \qquad (2)$$

$$\vec{n}\times\vec{H}_c = 0,\ over\ \partial_D\Omega_c \qquad (3)$$

Non-conducting region

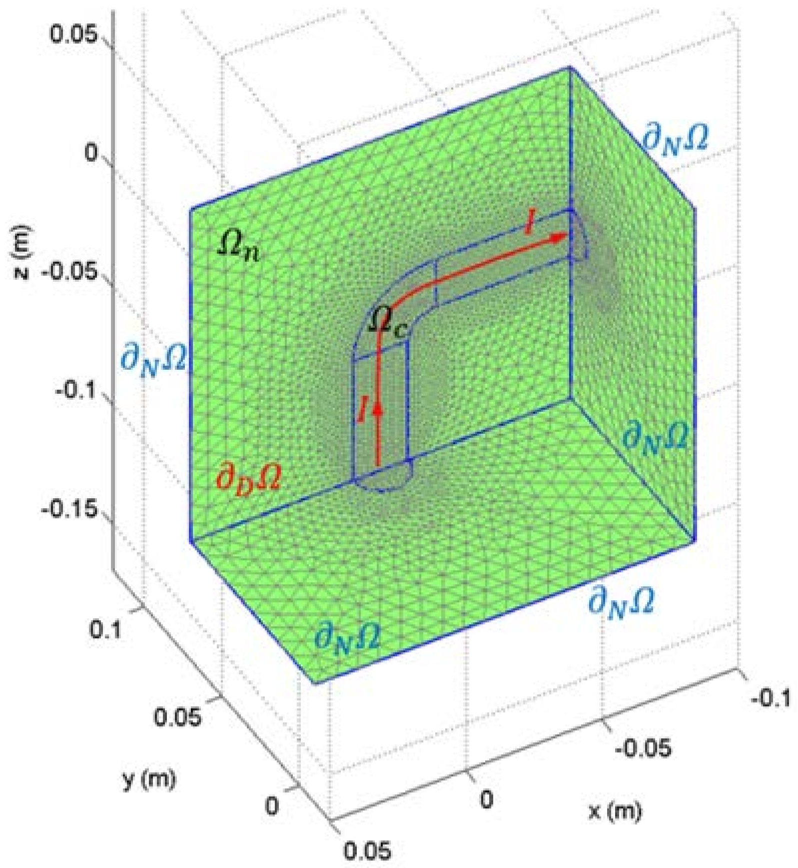
$$\vec{H}_n = \vec{H}_s - \nabla\Phi,\ in\ \Omega_n \qquad (4)$$

$$\frac{\partial\Phi}{\partial n} = \vec{n}\cdot\vec{H}_s,\ over\ \partial_N\Omega_n \qquad (5)$$

$$\Phi = 0,\ over\ \partial_D\Omega_n \qquad (6)$$

[4]J. P. Webb, B. Forghani, “T-Ω Method Using Hierarchical Edge Elements”, IEE Proceedings – Science, Measurements and Technology, Vol. 142, No. 2, pp. 133-141, 1995.

Scalar + Vector Finite Element Method (FEM), $\vec{H} - \Phi$ field formulation



$$\nabla \times \left(\frac{1}{j\omega\mu_0\sigma} \nabla \times \vec{H}_c \right) + \mu_r \vec{H}_c = 0, \; in \; \Omega_c$$

$$\nabla \left(\mu_r \nabla \Phi \right) = 0, \; in \; \Omega_n$$

$$\vec{n} \times \vec{H}_c = \vec{n} \times \vec{H}_s - \vec{n} \times \nabla \Phi, \; over \; \partial_{cn}\Omega$$

$$\mu_{rn} \frac{\partial \Phi}{\partial n} = \mu_{rn} \vec{n} \cdot \vec{H}_s - \mu_{rc} \vec{n} \cdot \vec{H}_c, \; over \; \partial_{cn}\Omega$$

$$\vec{n} \times \vec{H}_c = 0, \; over \; \partial_D \Omega_c$$

$$\vec{n} \times \left(\frac{1}{\sigma} \nabla \times \vec{H}_c \right) = 0, \; over \; \partial_N \Omega_c$$

$$\frac{\partial \Phi}{\partial n} = \vec{n} \cdot \vec{H}_s, \; over \; \partial_N \Omega_n$$

$$\Phi = 0, \; over \; \partial_D \Omega_n$$

[4]J. P. Webb, B. Forghani, “T-Ω Method Using Hierarchical Edge Elements”, IEE Proceedings – Science, Measurements and Technology, Vol. 142, No. 2, pp. 133-141, 1995.

Scalar + Vector Finite Element Method (FEM), $\vec{H} - \Phi$ field formulation

$$\begin{bmatrix} A^{(1)} + A^{(2)} + A^{(3)} & B \\ C & D \end{bmatrix} \begin{Bmatrix} H_{ec}^{c} \\ \Phi^{n} \end{Bmatrix} = \begin{Bmatrix} b^{c} \\ b^{n} \end{Bmatrix}$$

$$A_{ij}^{(1)} = \frac{1}{j\omega\mu_0} \sum_{\substack{e \\ i,j\in\Omega_c^e}} \iiint_{(\Omega_c^e)} \frac{1}{\sigma} \left(\nabla\times\vec{N}_i\right)\cdot\left(\nabla\times\vec{N}_j\right) dV$$

$$A_{ij}^{(2)} = \sum_{\substack{e \\ i,j\in\Omega^e}} \iiint_{(\Omega_c^e)} \mu_{rc} \vec{N}_i \cdot \vec{N}_j dV$$

$$A_{ij}^{(3)} = \frac{1}{j\omega\mu_0} \sum_{\substack{e \\ i,j\in\Delta_{cn}^e}} \iint_{(\Delta_{cn}^e)} \vec{N}_i \cdot \left[\vec{n}_c \times \left(\frac{1}{\sigma}\nabla\times\vec{N}_j\right)\right] dS$$

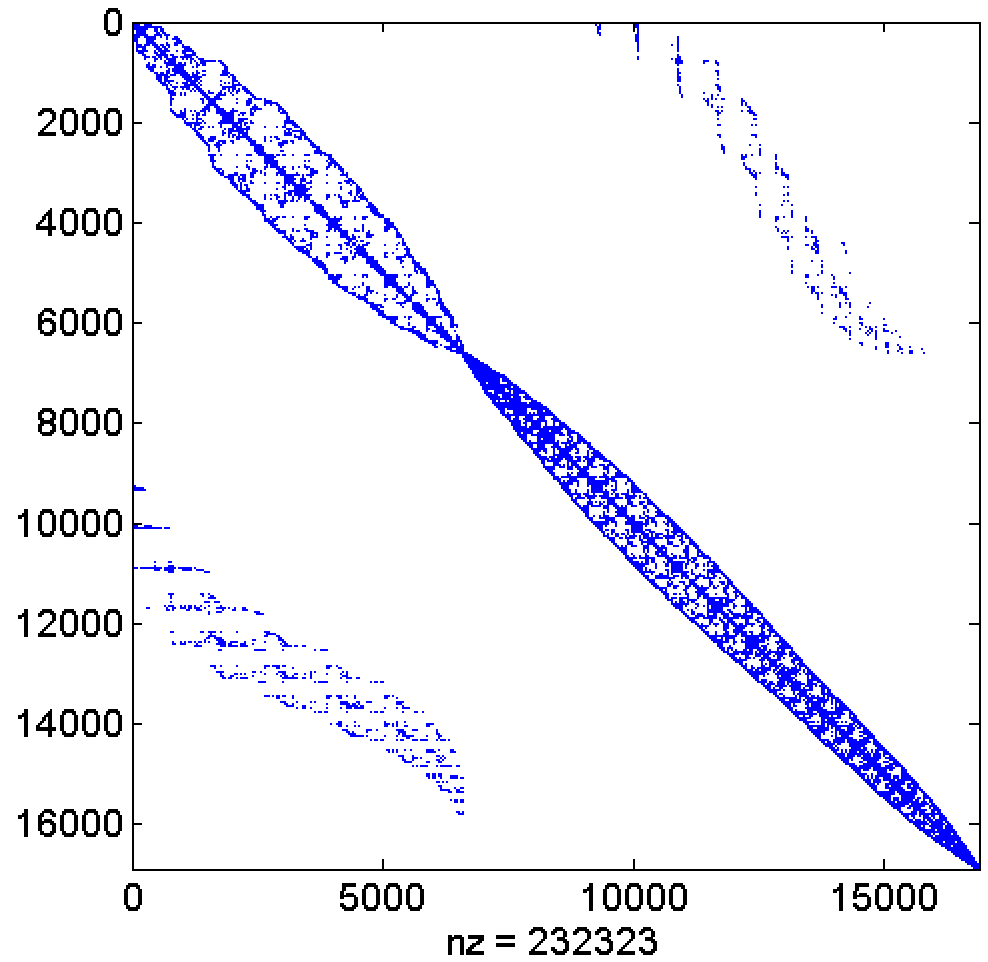
$$B_{ij} = \begin{cases} -1 & j = n_1^{(edge\ i)} \\ +1 & j = n_2^{(edge\ i)} \end{cases}, \quad K_{ij} = 0, K_{ii} = 1$$

$$b_i^c = 0$$

$$D_{ij} = \sum_{\substack{e \\ i,j\in\Omega_n^e}} \iiint_{(\Omega_n^e)} \mu_{rn} \nabla N_i \cdot \nabla N_j \, dV$$

$$C_{ij} = -\sum_{\substack{e \\ i,j\in\Delta_{cn}^e}} \iint_{(\Delta_{cn}^e)} N_i \, \mu_{rc} \vec{n}_c \cdot \vec{N}_j \, dS$$

$$b_i^n = \sum_{\substack{e \\ i\in\Delta_{nN}^e}} \iint_{(\Delta_{nN}^e)} \mu_{rn} N_i \, \vec{n}_n \cdot \vec{H}_s \, dS$$



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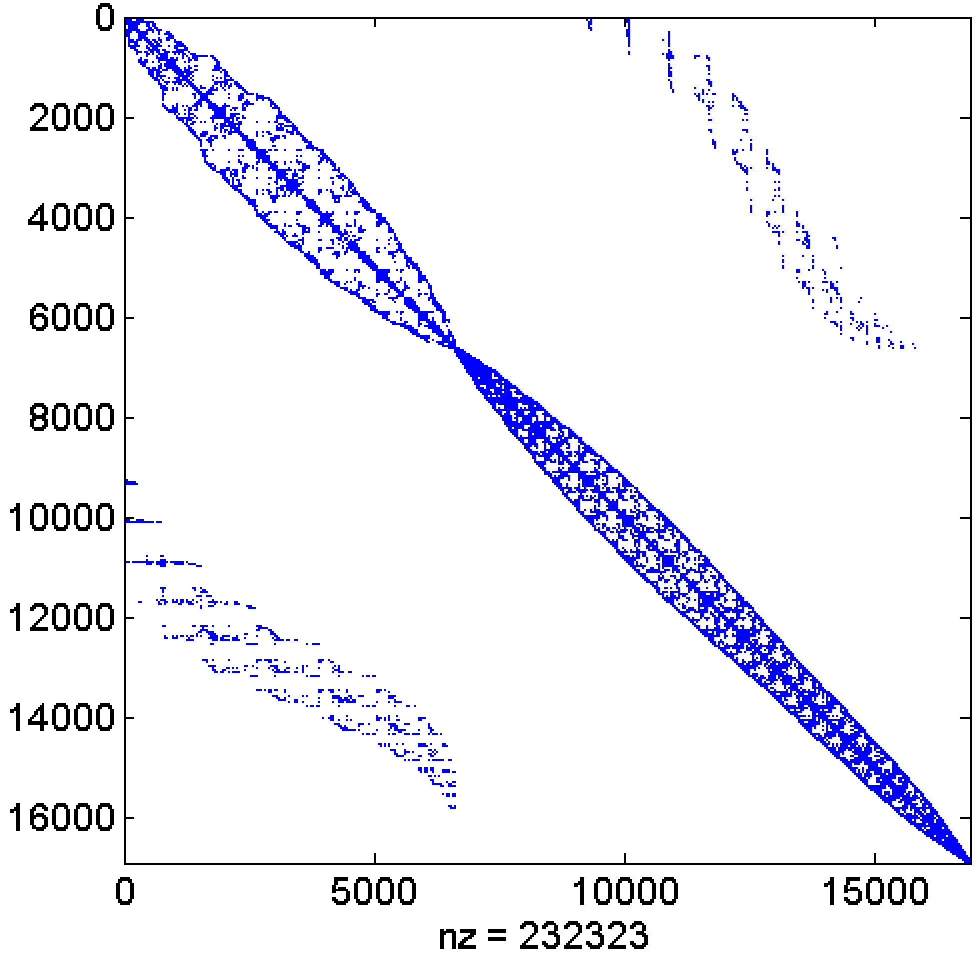
Scalar + Vector Finite Element Method (FEM), $\vec{H} - \Phi$ field formulation

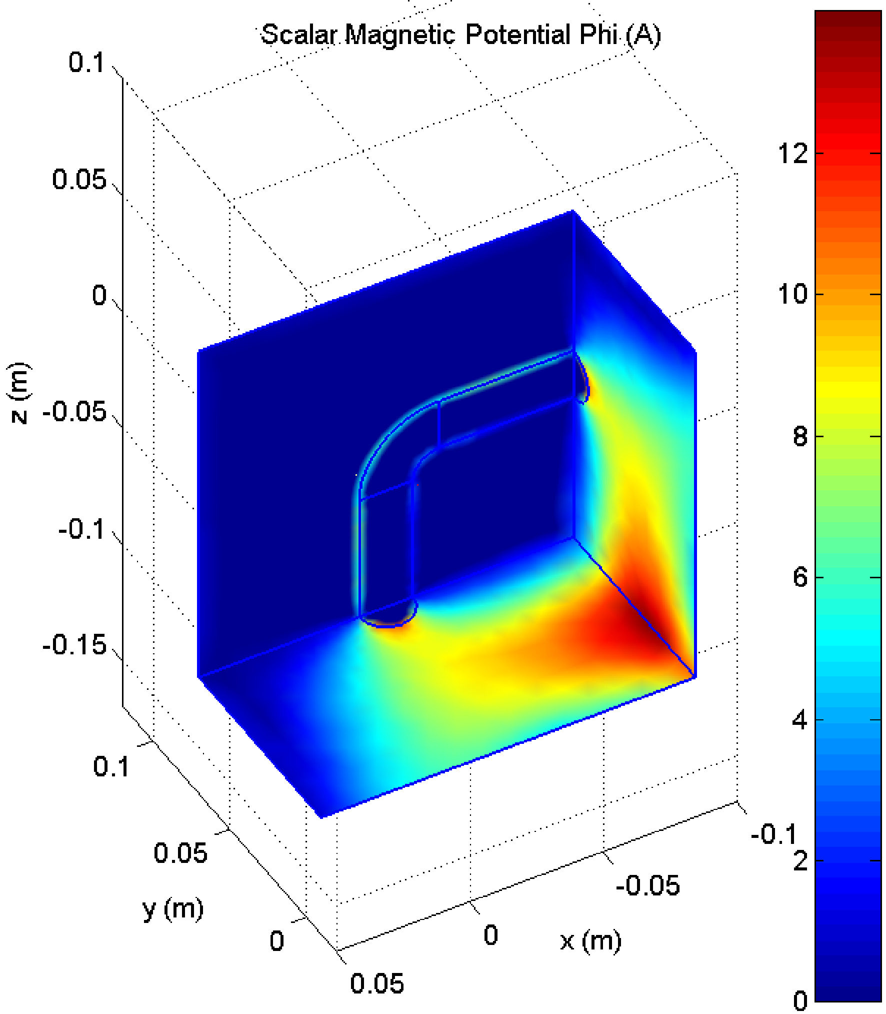


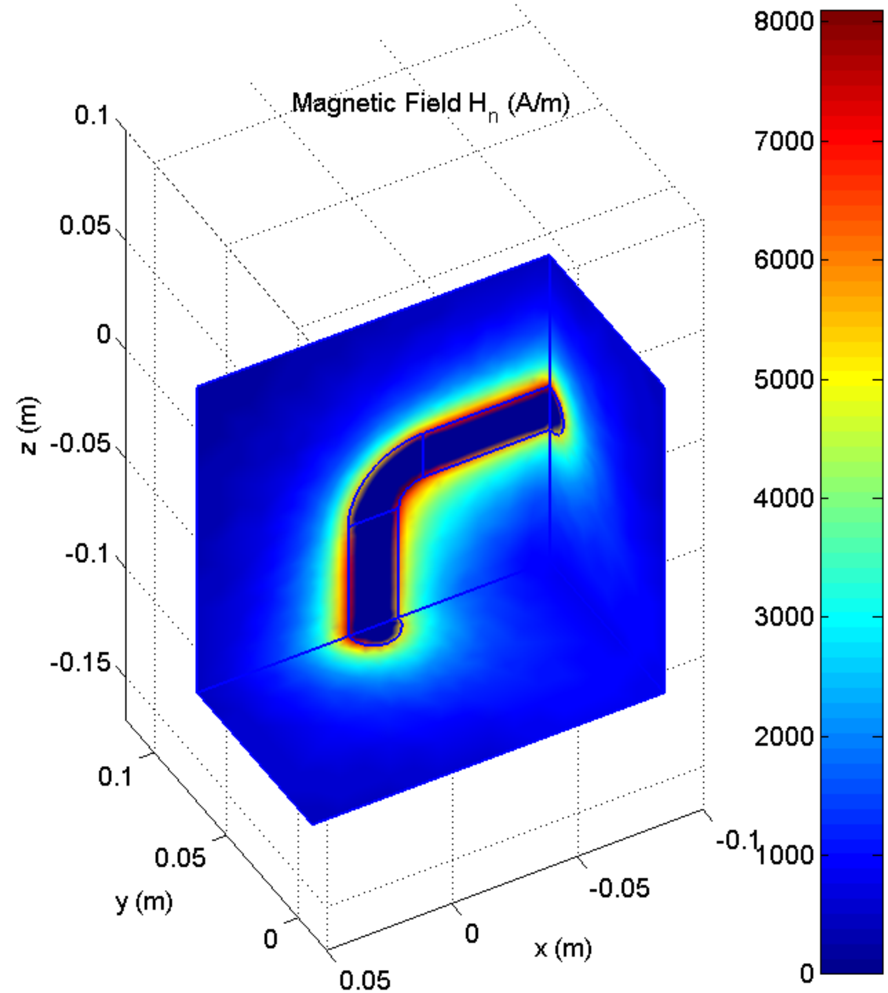
N_{eq}=16’913

$$Density = \frac{N_{nz}}{N_{eq}^2} \cdot 100\% = 0.081\%$$

$$\kappa(K) = \|K\| \cdot \|K^{-1}\| = 1.48 \cdot 10^{15}$$

Scalar + Vector Finite Element Method (FEM), $\vec{H} - \Phi$ field formulation





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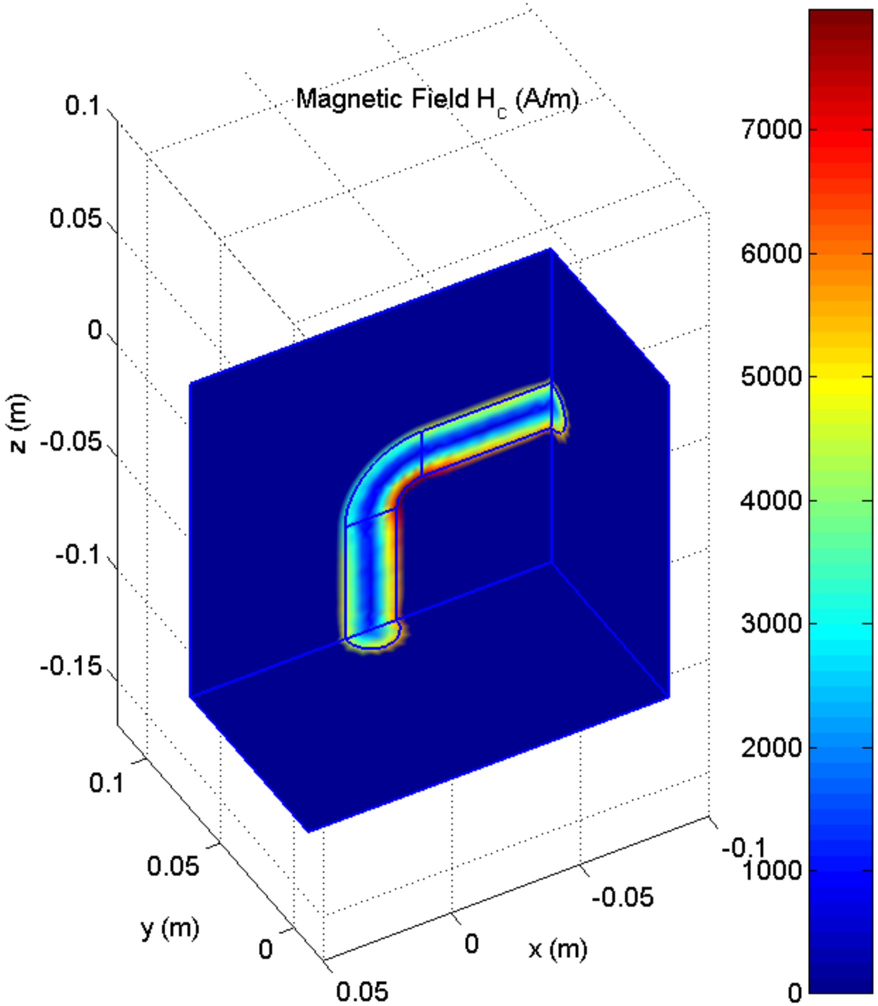
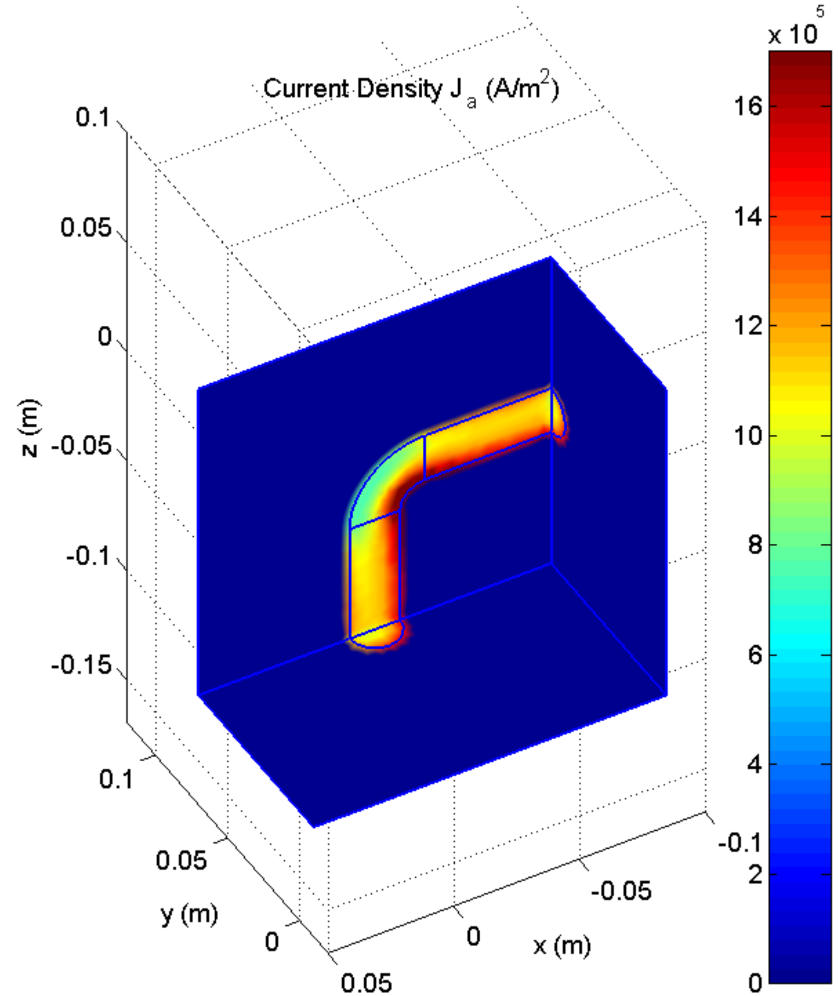
Scalar + Vector Finite Element Method (FEM), $\vec{H} - \Phi$ field formulation





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Boundary Element Method (BEM), $\vec{H}-\Phi$ field formulation

$$-\frac{1}{2}\vec{J}(\xi)+\frac{1}{4\pi}\oiint_{(\partial\Omega)}\left[\vec{n}_\xi\times\left(\vec{J}(\eta)\times\nabla_\xi K(\eta,\xi)\right)\right]dS_\eta -$$

$$-\frac{1}{4\pi}\oiint_{(\partial\Omega)}\left[\sigma_m(\eta)\left(\vec{n}_\xi\times\nabla_\xi G(\eta,\xi)\right)\right]dS_\eta = -\left[\vec{H}_0^t(\xi)+\vec{H}_\delta^t(\xi)\right]$$

$$\mathrm{G}(\eta,\xi)=\frac{1}{\mathrm{r}_{\eta,\xi}} \qquad \forall\,\eta,\xi\in\partial\Omega$$

$$-\frac{1}{2}\sigma_m(\xi)-\frac{1}{4\pi}\oiint_{(\partial\Omega)}\left[\sigma_m(\eta)\left(\vec{n}_\xi\cdot\nabla_\xi G(\eta,\xi)\right)\right]dS_\eta -$$

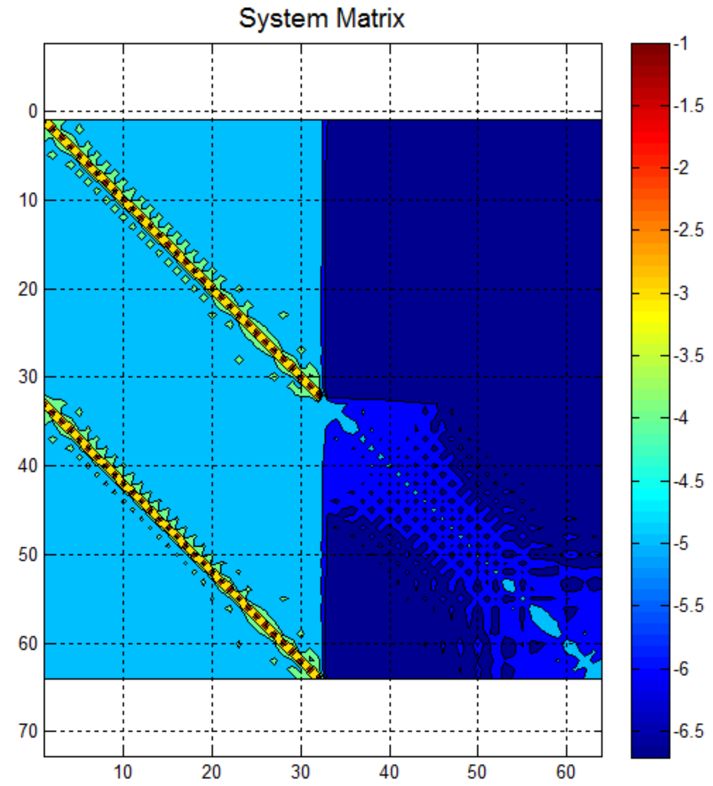
$$-\frac{\mu}{4\pi\mu_0}\oiint_{(\partial\Omega)}\left[\vec{n}_\xi\cdot\left(\vec{J}(\eta)\times\nabla_\xi K(\eta,\xi)\right)\right]dS_\eta = -\left[\vec{H}_0^n(\xi)+\vec{H}_\delta^n(\xi)\right]$$

$$\mathrm{K}(\eta,\xi)=\frac{\mathrm{e}^{-(1+\mathrm{j})\cdot\mathrm{k}\cdot\mathrm{r}_{\eta,\xi}}}{\mathrm{r}_{\eta,\xi}} \qquad \mathrm{k}=\sqrt{\omega\mu_0\mu_r\sigma/2}$$

$\vec{J}$ – virtual current σ_m – virtual magnetic charge

J. Smajic, et al., “**BEM-based Simulations in Engineering Design**,” in “Boundary Element Analysis: Mathematical Aspects and Applications,” (Edited by M. Schanz and O. Steinbach) Lecture Notes in Applied and Computational Mechanics, Vol. 29, pp. 281-352, **Springer Verlag**, Berlin, Heidelberg, New York, 2007.

Boundary Element Method (BEM), $\vec{H} - \Phi$ field formulation



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Boundary Element Method (BEM), $\vec{H} - \Phi$ field formulation

Kernel expansion: Fast Multipole Technique (FMT)

$$|x-y| >> 0 \,(\, farfield \,) \Rightarrow$$

$$K(x,y) \approx K_m(x,y;x_0,y_0) = \sum_{(\mu,\nu)\in I_m} K_{(\mu,\nu)}(x_0,y_0)\cdot X_\mu(x,x_0)\cdot Y_\nu(y,y_0)$$

Taylor-, Multipole-, Cebysev- expansion

$$|x-x_0| + |y-y_0| \le \eta \cdot |x_0 - y_0|$$

Far-field condition

GMRES with clustering

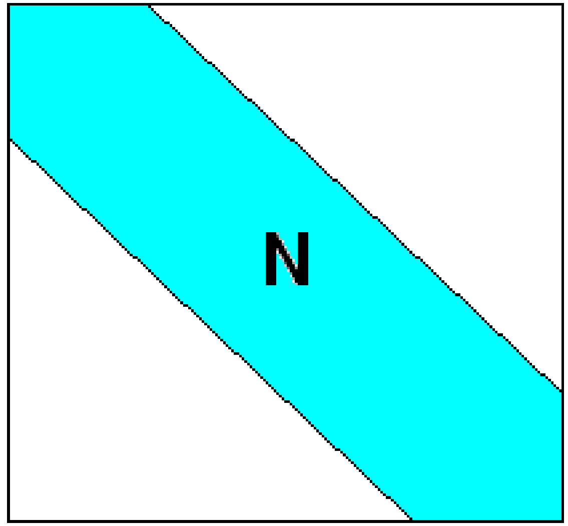
$$v = \tilde{A}\cdot u = N\cdot u + \sum_{(\sigma,\tau)\in F} X_\sigma^T\left(F_{\sigma\tau}\left(Y_\tau \cdot u\right)\right)$$

Matrix-vector multiplication

J. Smajic, et al., “**BEM-based Simulations in Engineering Design**,” in “Boundary Element Analysis: Mathematical Aspects and Applications,” (Edited by M. Schanz and O. Steinbach) Lecture Notes in Applied and Computational Mechanics, Vol. 29, pp. 281-352, **Springer Verlag**, Berlin, Heidelberg, New York, 2007.

Boundary Element Method (BEM), $\vec{H} - \Phi$ field formulation

$\tilde{A} =$



GMRES with clustering

$$v = \tilde{A} \cdot u = N \cdot u + \sum_{(\sigma,\tau)\in F} X_{\sigma}^{T} \left(F_{\sigma\tau} \left(Y_{\tau} \cdot u \right) \right)$$

Matrix-vector multiplication

J. Smajic, et al., “**BEM-based Simulations in Engineering Design**,” in “Boundary Element Analysis: Mathematical Aspects and Applications,” (Edited by M. Schanz and O. Steinbach) Lecture Notes in Applied and Computational Mechanics, Vol. 29, pp. 281-352, **Springer Verlag**, Berlin, Heidelberg, New York, 2007.

Boundary Element Method (BEM), $\vec{H} - \Phi$ field formulation

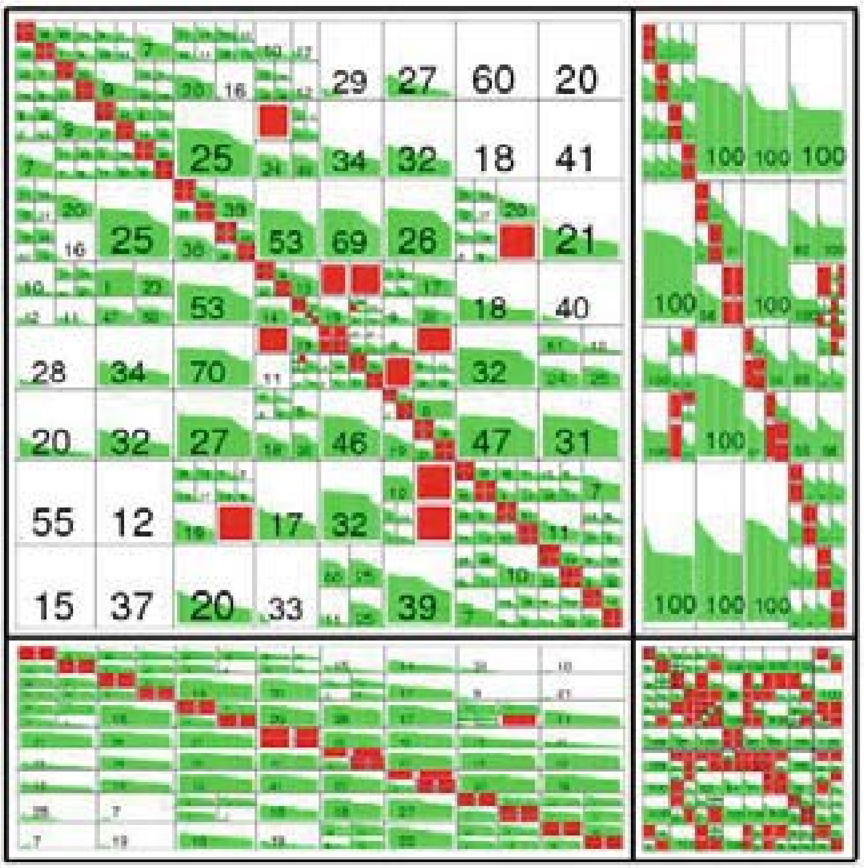
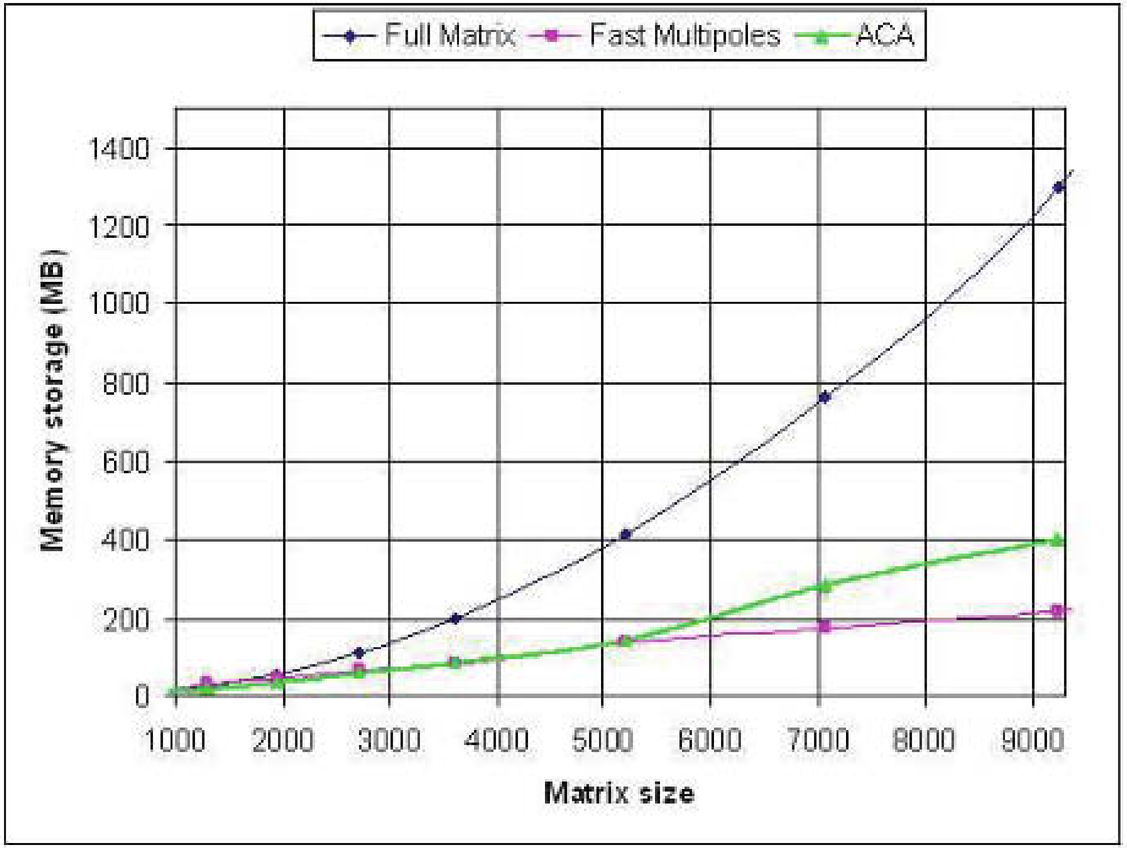


Fig. 40. The rank distribution over the blocks of a typical BEM matrix approximated by H-matrices and ACA (an eddy-current example from the Table 1 with 9224 DOFs).

J. Smajic, Z. Andjelic, M. Bebendorf, “Fast BEM for Eddy-Current Problems Using H-matrices and Adaptive Cross Approximation”, IEEE Transactions on Magnetics, Vol. 43, Issue 4, , pp. 1269 -1272, April 2007.

Boundary Element Method (BEM), $\vec{H} - \Phi$ field formulation



J. Smajic, Z. Andjelic, M. Bebendorf, “Fast BEM for Eddy-Current Problems Using H-matrices and Adaptive Cross Approximation”, **IEEE Transactions on Magnetics**, Vol. 43, Issue 4, , pp. 1269 -1272, April 2007.

Coupled FEM - MMP

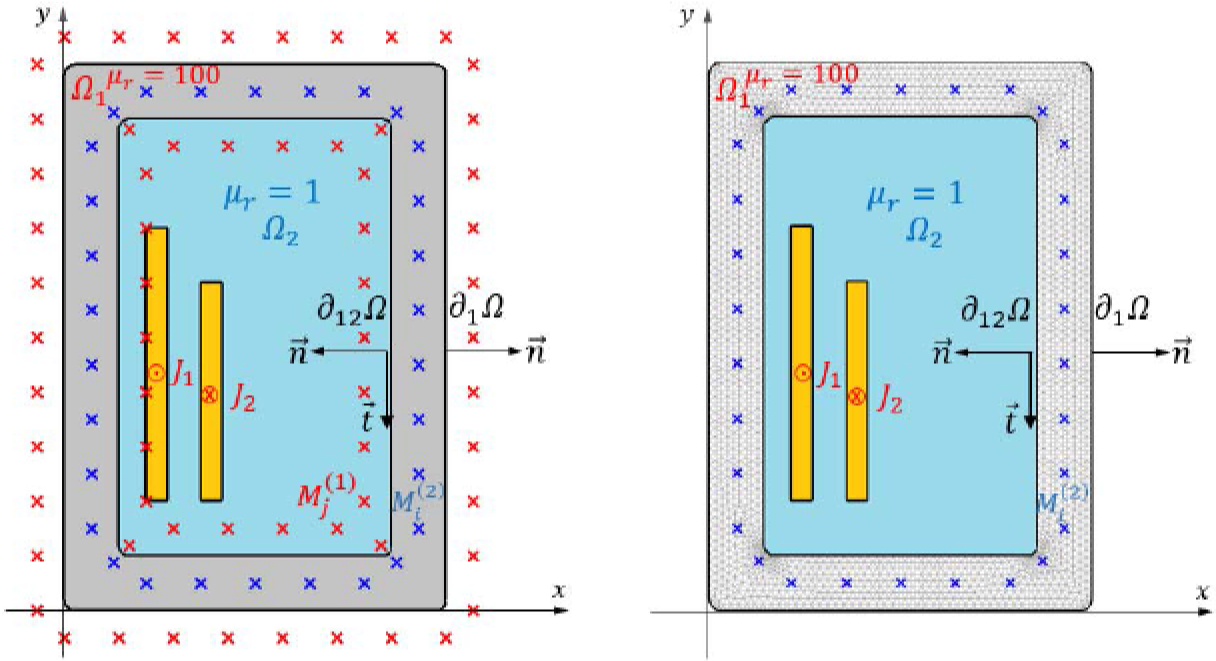
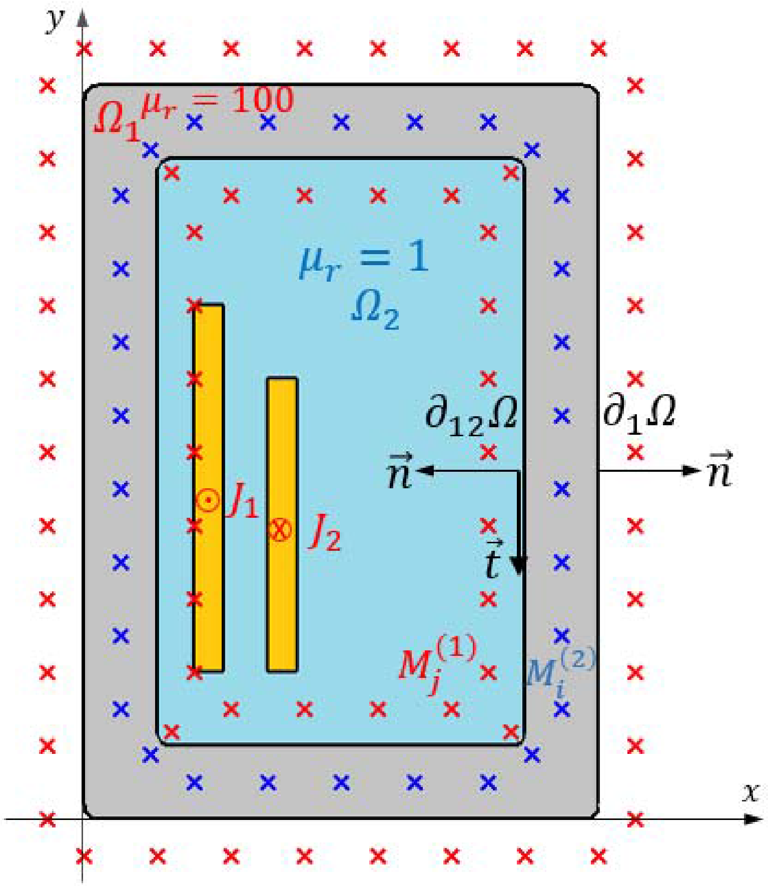


Fig. 1. Simple 2-D MS problem for demonstrating the MMP technique (left) and coupled FEM–MMP (right) is shown. The field sources are the windings (orange region) with known current densities (J_1 and J_2). A ferromagnetic core (gray region) Ω_1 surrounds the air window (blue region) Ω_2 ($\mu_r = 1$).

J. Smajic, Ch. Hafner, J. Leuthold, "Coupled FEM-MMP for Computational Electromagnetics", **IEEE Transactions on Magnetics**, Vol. 52, No. 3, pp. 7207704, March 2016.

Coupled FEM - MMP



$$\Omega_1: \; A_{z1}(x,y) = \sum_{p=1}^{N_{\text{me}}^{(1)}} \left\{ B_0^{p(1)} \cdot \ln r_p + \sum_{k=1}^{m_p} \frac{1}{r_p^k} \left[B_k^{p(1)} \cdot \cos(k\varphi_p) + D_k^{p(1)} \cdot \sin(k\varphi_p) \right] \right\} \quad (3)$$

$$\Omega_2: \; A_{z2}(x,y) = A_{zs} + \sum_{p=1}^{N_{\text{me}}^{(2)}} \left\{ B_0^{p(2)} \cdot \ln r_p + \sum_{k=1}^{m_p} \frac{1}{r_p^k} \left[B_k^{p(2)} \cdot \cos(k\varphi_p) + D_k^{p(2)} \cdot \sin(k\varphi_p) \right] \right\} \quad (4)$$

$$\partial_{12\otimes}\Omega: \; \vec{n} \times \vec{H}_1 = \vec{n} \times \vec{H}_2 \Rightarrow \vec{t} \cdot \vec{H}_1 = \vec{t} \cdot \vec{H}_2 \quad (5)$$

$$\partial_{12\odot}\Omega: \; \vec{n} \cdot \vec{B}_1 = \vec{n} \cdot \vec{B}_2 \Rightarrow A_{z1} = A_{z2} \quad (6)$$

$$\partial_{1\odot}\Omega: \; \vec{n} \cdot \vec{B}_1 = 0 \Rightarrow A_{z1} = 0 \quad (7)$$

J. Smajic, Ch. Hafner, J. Leuthold, “Coupled FEM-MMP for Computational Electromagnetics”, **IEEE Transactions on Magnetics**, Vol. 52, No. 3, pp. 7207704, March 2016.

Numerical Methods for Computing 3-D Vector Fields

Coupled FEM - MMP

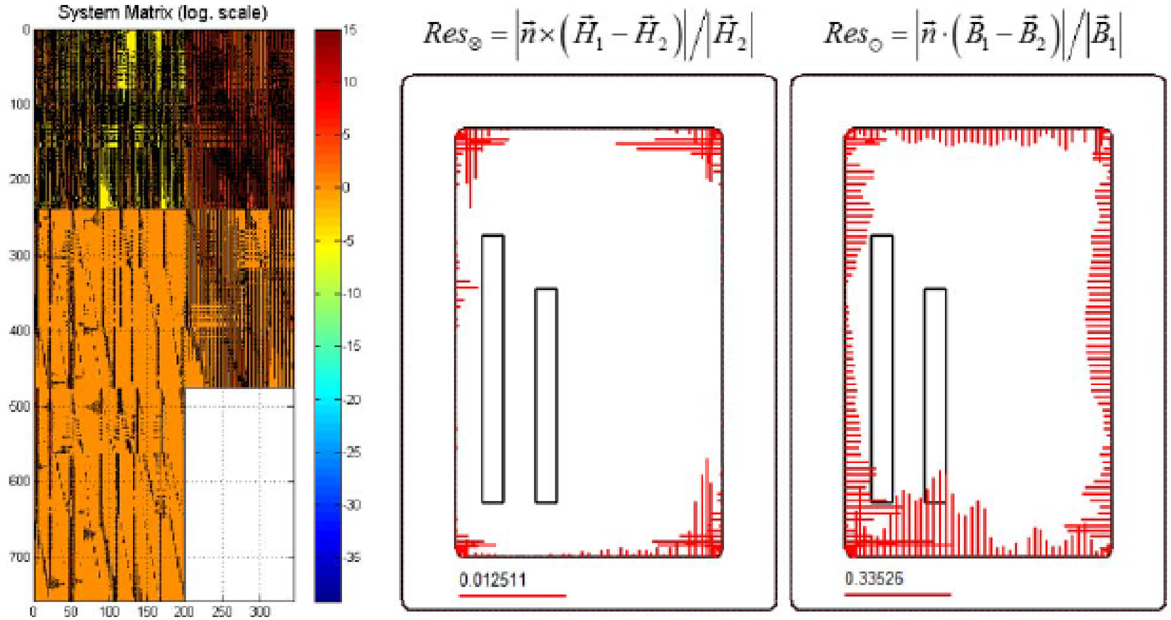
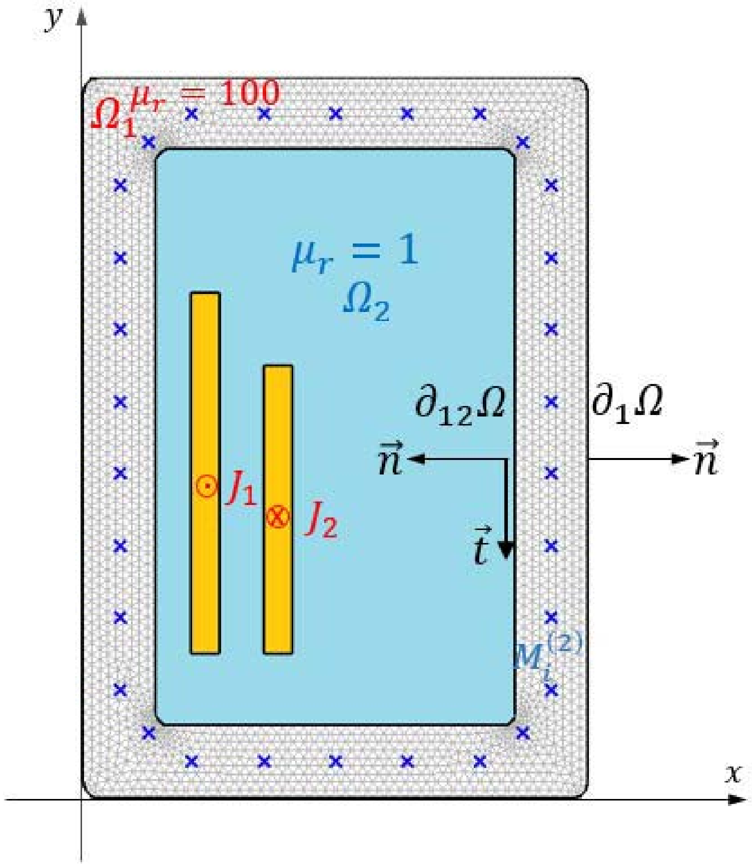


Fig. 2. Left: MMP matrix of the chosen MS problem. Different blocks of the equation system (8) are noticeable. One empty block (bottom right) related to (7) is visible. Right: Relative residual of the H-field and B-field over the interface boundary obtained from MMP, with the horizontal line on the bottom as a reference.

J. Smajic, Ch. Hafner, J. Leuthold, “Coupled FEM-MMP for Computational Electromagnetics”, **IEEE Transactions on Magnetics**, Vol. 52, No. 3, pp. 7207704, March 2016.

Coupled FEM - MMP



$$\Omega_1 : \oint_{(\partial_{12}\Omega)} \frac{1}{\mu_1} N_i \nabla A_{z1} \cdot \vec{n}\, dl - \iint_{(\Omega_1)} \frac{1}{\mu_1} \nabla N_i \cdot \nabla A_{z1} dS = 0$$

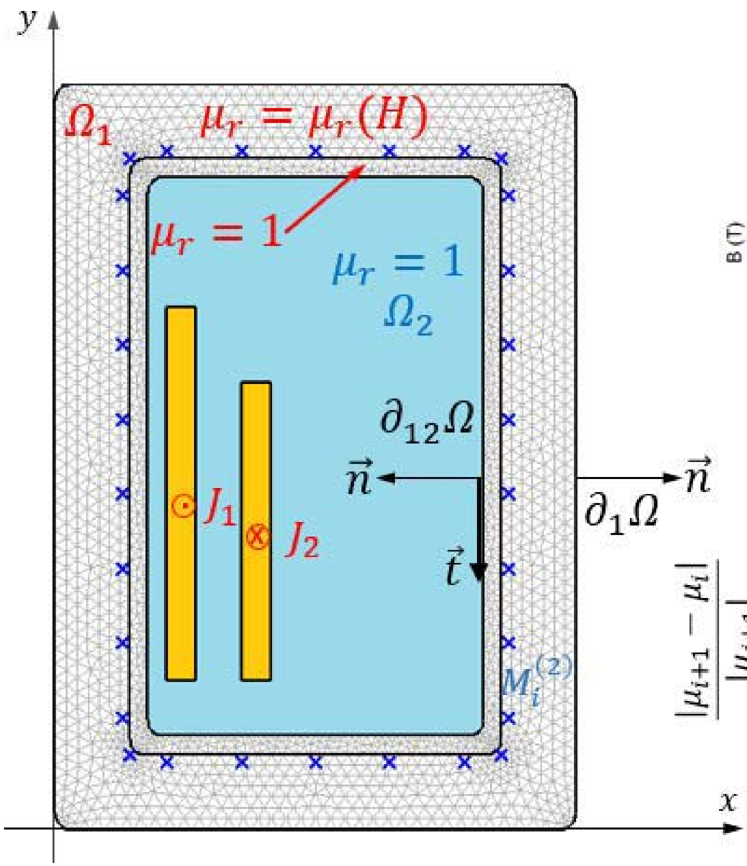
$$\oint_{(\partial_{12}\Omega)} N_i \vec{t} \cdot \vec{H}_{2m} dl + \iint_{(\Omega_2)} \frac{1}{\mu_1} \nabla N_i \cdot \nabla A_{z1} dS = - \oint_{(\partial_{12}\Omega)} N_i \vec{t} \cdot \vec{H}_s dl$$

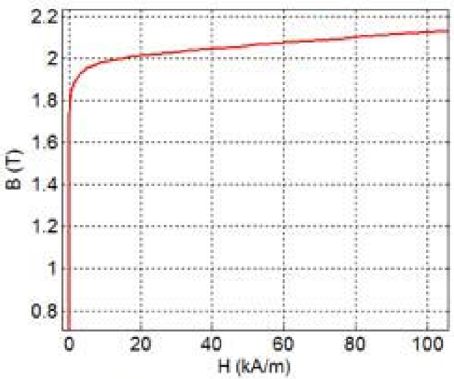
$$\partial_{12}\Omega : \; A_{z1} - \sum_{p=1}^{N_{\mathrm{me}}} \Big\{ B_0^p \cdot \ln r_p + \sum_{k=1}^{m_p} \frac{1}{r_p^k} \left[B_k^p \cdot \cos(k\varphi_p) + D_k^p \cdot \sin(k\varphi_p) \right] \Big\} = A_{zs}.$$

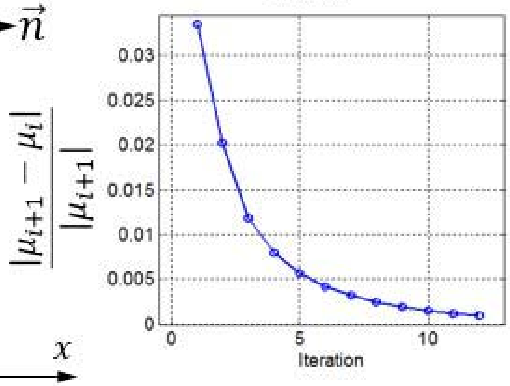
$$\begin{bmatrix} A_{\mathrm{FEM}} & C_{\mathrm{FM}\otimes} \\ C_{\mathrm{MF}\odot} & B_{\mathrm{MMP}} \end{bmatrix} \cdot \begin{Bmatrix} x_{\mathrm{FEM}} \\ x_{\mathrm{MMP}} \end{Bmatrix} = \begin{Bmatrix} b_\otimes \\ b_\odot \end{Bmatrix}$$

J. Smajic, Ch. Hafner, J. Leuthold, “Coupled FEM-MMP for Computational Electromagnetics”, **IEEE Transactions on Magnetics**, Vol. 52, No. 3, pp. 7207704, March 2016.

Coupled FEM - MMP







J. Smajic, Ch. Hafner, J. Leuthold, "Coupled FEM-MMP for Computational Electromagnetics", **IEEE Transactions on Magnetics**, Vol. 52, No. 3, pp. 7207704, March 2016.

Coupled FEM - MMP

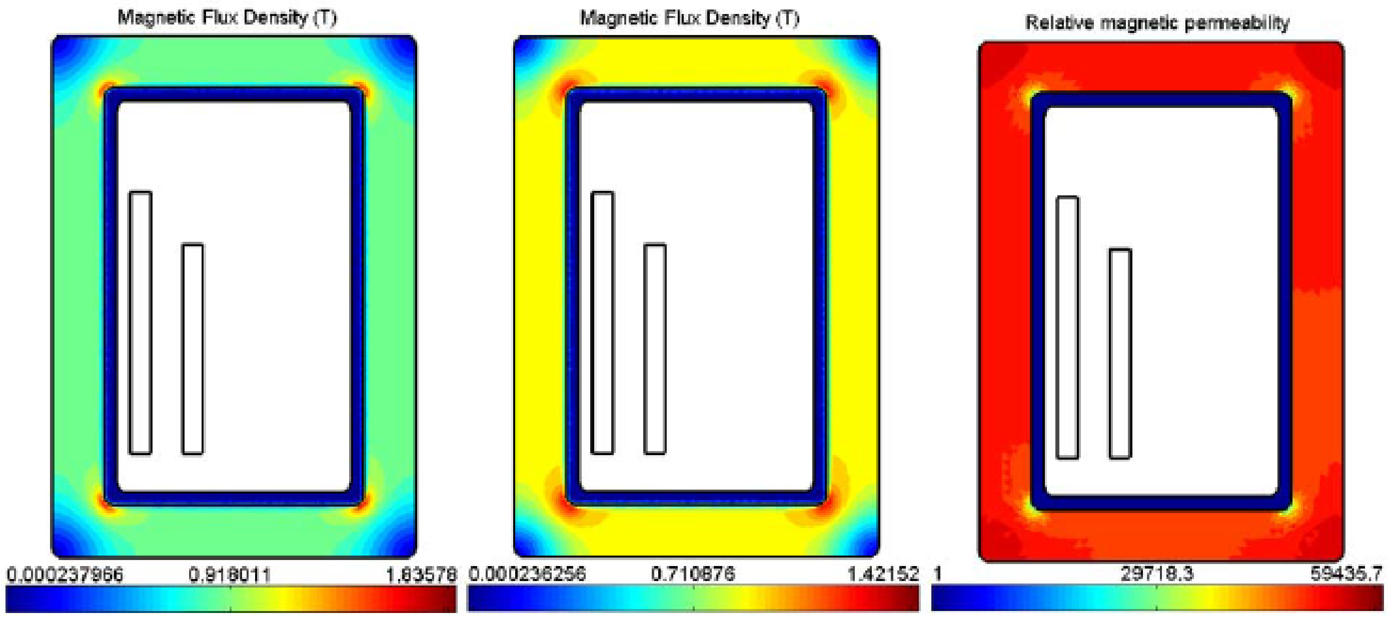


Fig. 7. Results of the linear (left) and nonlinear (middle and right) FEM–MMP analysis are presented. The regions of the magnetic saturation are visible in the nonlinear results. They are slightly asymmetric due to the asymmetry of the winding system.

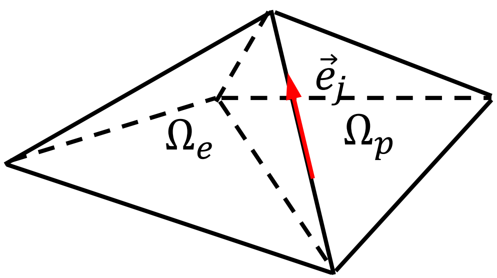
J. Smajic, Ch. Hafner, J. Leuthold, "Coupled FEM-MMP for Computational Electromagnetics", **IEEE Transactions on Magnetics**, Vol. 52, No. 3, pp. 7207704, March 2016.

Discontinuous Galerkin Time-domain FEM

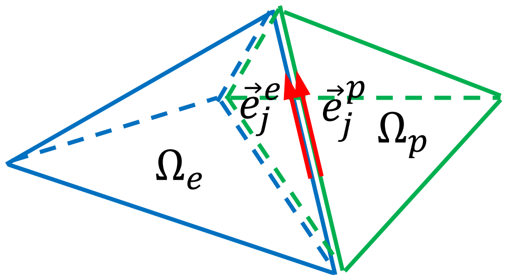
$$\iiint\limits_{(\Omega_e)} \vec{N}_i \cdot \left(\nabla \times \vec{E}\right) dV + \iiint\limits_{(\Omega_e)} \mu \vec{N}_i \cdot \frac{\partial \vec{H}}{\partial t} dV = -\frac{1}{2} \oiint\limits_{(\partial\Omega_e)} \vec{N}_i \cdot \left(\vec{n} \times \vec{E}^+ - \vec{n} \times \vec{E}\right) dS \quad (12)$$

$$\iiint\limits_{(\Omega_e)} \vec{N}_i \cdot \left(\nabla \times \vec{H}\right) dV - \iiint\limits_{(\Omega_e)} \varepsilon \vec{N}_i \cdot \frac{\partial \vec{E}}{\partial t} dV = -\frac{1}{2} \oiint\limits_{(\partial\Omega_e)} \vec{N}_i \cdot \left(\vec{n} \times \vec{H}^+ - \vec{n} \times \vec{H}\right) dS \quad (13)$$

Continuous Galerkin



Discontinuous Galerkin



Discontinuous Galerkin Time-domain FEM

$$\left\{H_e^{\left(k+\frac{1}{2}\right)}\right\} = \left\{H_e^{\left(k-\frac{1}{2}\right)}\right\} - \Delta t\left[A_e^2\right]^{-1}\left(\left[A_e^1\right]\left\{E_e^{(k)}\right\} + \frac{1}{2}\sum_{f=1}^{4}\left[\left[B_{ep}^1\right]\left\{E_e^{+(k)}\right\} - \left[B_{ee}^1\right]\left\{E_e^{(k)}\right\}\right]\right) \quad (21)$$

$$\left\{E_e^{(k+1)}\right\} = \left\{E_e^{(k)}\right\} + \Delta t\left[A_e^3\right]^{-1}\left(\left[A_e^1\right]\left\{H_e^{\left(k+\frac{1}{2}\right)}\right\} + \frac{1}{2}\sum_{f=1}^{4}\left[\left[B_{ep}^1\right]\left\{H_e^{+\left(k+\frac{1}{2}\right)}\right\} - \left[B_{ee}^1\right]\left\{H_e^{\left(k+\frac{1}{2}\right)}\right\}\right]\right) \quad (22)$$

Matrix entries have the following form

$$A_e^1(i,j) = \iiint_{(\Omega_e)} \vec{N}_i \cdot \left(\nabla \times \vec{N}_j\right) dV \quad (23)$$

$$A_e^2(i,j) = \iiint_{(\Omega_e)} \mu \vec{N}_i \cdot \vec{N}_j dV \quad (24)$$

$$A_e^3(i,j) = \iiint_{(\Omega_e)} \varepsilon \vec{N}_i \cdot \vec{N}_j dV \quad (25)$$

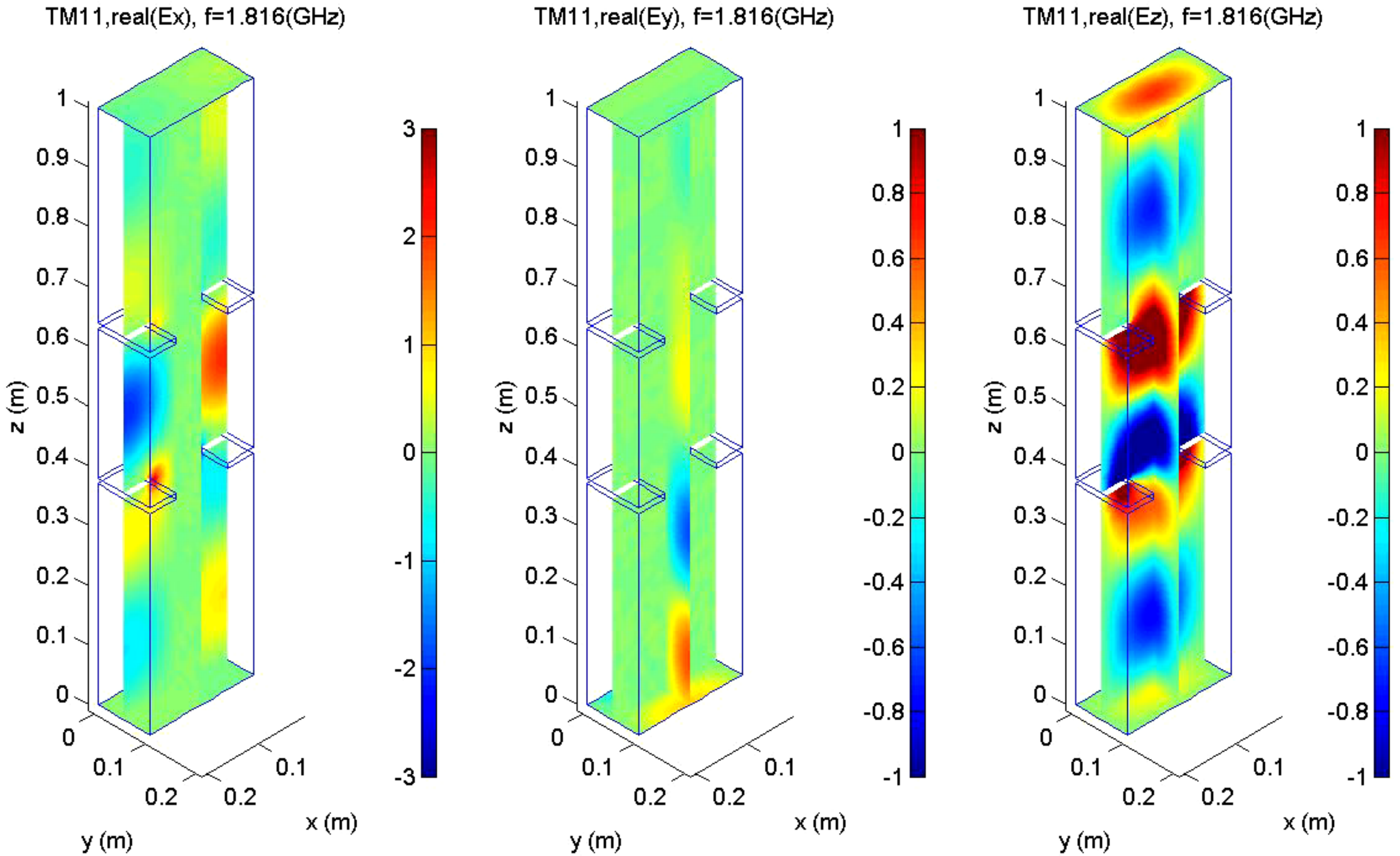
$$B_{ep}^1(f,i,j) = \iint_{(\Delta_e^f)} \vec{N}_i \cdot \left(\vec{n} \times \vec{N}_j^+\right) dS \quad (26)$$

$$B_{ee}^1(f,i,j) = \iint_{(\Delta_e^f)} \vec{N}_i \cdot \left(\vec{n} \times \vec{N}_j\right) dS \quad (27)$$

Heuristic stability condition

$$\Delta t = \frac{\min\left(\frac{h_l}{3}\sqrt{\mu_r \varepsilon_r}\,\frac{1}{p^2}\right)}{c} \quad (28)$$

Discontinuous Galerkin Time-domain FEM



Summary

- FEM: sparse matrices, ill-conditioned matrices, efficiency depends on field formulation.
- BEM: dense matrices, matrix compression and preconditioning, singular integrals.
- MMP: boundary methods, sources away from boundary, no singular integrals, control of accuracy.
- FEM-MMP: possibility to solve nonlinear problems without air-box.
- DG-TD-FEM: no large linear system to solve, leapfrog time-stepping scheme, conditionally stable.