Rješenja prvog međuispita iz Matematike 3E i 3R

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1. (3 boda)

a) (**1b**) Funkcija $f : [a, b] \to \mathbf{R}$ zadovoljava Dirichletove uvjete na intervalu $[a, b]$, ako vrijedi

1) f je po dijelovima neprekinuta i njezini su prekidi prve vrste,

2) f je monotona ili ima najviše konačan broj strogih ekstrema.

b) (**1b**) Funkcija $f(x) = \frac{1}{1-x}$ ne zadovoljava Dirichletove uvjete na segmentu $[0, 2]$ jer u točki 1 ima prekid koji nije prekid prve vrste.

c) (**1b**) **Teorem o konvergenciji Fourierovog reda.**

Neka je f po dijelovima glatka periodična funkcija s periodom 2π koja zadovoljava Dirichletove uvjete. Tada njezin Fourierov red konvergira u svakoj točki $x \in [-\pi, \pi]$ i za sumu $S(x)$ reda vrijedi:

(i) $S(x) = f(x)$, ako je f neprekinuta u točki x

(ii) $S(x) = \frac{1}{2}[f(x - 0) + f(x + 0)]$, ako je x točka prekida za f.

2. (3 boda)

a) (**2b**) Razvijamo u red parno proširenje funkcije f. $T = 2$, $L = 1$, $b_n = 0$.

$$a_0 = \frac{2}{1}\left[\int_0^{\frac{1}{2}} 1\,dx + \int_{\frac{1}{2}}^{1} (-1)\,dx\right] = 2\left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

$$a_n = \frac{2}{1}\left[\int_0^{\frac{1}{2}} \cos(n\pi x)\,dx + \int_{\frac{1}{2}}^{1} (-1)\cos(n\pi x)\,dx\right] = 2\left[\frac{\sin n\pi x}{n\pi}\Big|_{x=0}^{\frac{1}{2}} - \frac{\sin n\pi x}{n\pi}\Big|_{x=\frac{1}{2}}^{1}\right] = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Uočavamo

$$\Rightarrow a_{2k} = 0, \quad a_{2k+1} = \frac{4}{(2k+1)\pi}\cdot(-1)^k$$

pa je traženi trigonometrijski Fourierov red

$$f(x) = \sum_{k=0}^{\infty} \frac{4}{\pi}\cdot\frac{(-1)^k}{2k+1}\cos[(2k+1)\pi x].$$

b) (**1b**) Parsevalova jednakost neposredno daje:

$$\sum_{k=0}^{\infty}\left(\frac{4}{\pi(2k+1)}\right)^2 = \frac{2}{2}\int_{-1}^{1} 1^2\,dx \Rightarrow \frac{16}{\pi^2}\cdot s = 2 \Rightarrow s = \frac{\pi^2}{8},$$

gdje smo sa s označili traženu sumu.

3. (4 boda)
$f(x)$ je očito parna funkcija,

$$\Rightarrow B(\lambda)=0, \quad f(x)=\int_0^\infty A(\lambda)\cos(\lambda x)\,d\lambda$$

$$\begin{aligned}
A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(\xi)\cos(\lambda\xi)\,d\xi \\
&= \frac{2}{\pi}\int_0^1 (1-\xi)\cos(\lambda\xi)\,d\xi \\
&= \frac{2}{\pi}\int_0^1 \cos(\lambda\xi)\,d\xi - \frac{2}{\pi}\int_0^1 \underbrace{\xi}_{u}\underbrace{\cos(\lambda\xi)\,d\xi}_{dv} \\
&= \frac{2}{\pi}\cdot\frac{\sin(\lambda\xi)}{\lambda}\bigg|_{\xi=0}^1 - \frac{2}{\pi}\cdot\left[\xi\frac{\sin(\lambda\xi)}{\lambda}\bigg|_{\xi=0}^1 - \int_0^1 \frac{\sin(\lambda\xi)}{\lambda}\,d\xi\right] \\
&= \frac{2}{\pi}\cdot\frac{\sin\lambda}{\lambda} - \frac{2}{\pi}\left[\frac{\sin\lambda}{\lambda} + \frac{\cos(\lambda\xi)}{\lambda^2}\bigg|_{\xi=0}^1\right] \\
&= \frac{2}{\pi\lambda^2}(1-\cos\lambda)
\end{aligned}$$

Fourierov integral zadane funkcije f je:

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\cos(\lambda x)\,d\lambda.$$

Ako uvrstimo točku $x=0$, iz

$$f(0)=1\Rightarrow 1=\frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\,d\lambda=\frac{2}{\pi}\int_0^\infty \frac{2\sin^2\left(\frac{\lambda}{2}\right)}{\lambda^2}\,d\lambda=\frac{2}{\pi}\int_0^\infty \frac{\sin^2\left(\frac{\lambda}{2}\right)}{\left(\frac{\lambda}{2}\right)^2}\,d\left(\frac{\lambda}{2}\right)$$

$$\Rightarrow \left|\frac{\lambda}{2}=x\right| \Rightarrow 1=\frac{2}{\pi}\int_0^\infty \frac{\sin^2 x}{x^2}\,dx \Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}$$

4. (3 boda)

$$\cos t \quad \circ\!\!-\!\!\bullet \quad \frac{s}{s^2+1}$$

$$f(t) := t^2 \cos t \quad \circ\!\!-\!\!\bullet \quad (-1)^2 \left(\frac{s}{s^2+1}\right)'' = \left(\frac{1-s^2}{(s^2+1)^2}\right)' = \cdots = \frac{2s\cdot(s^2-3)}{(s^2+1)^3} =: F(s)$$

$$f(t) \quad \circ\!\!-\!\!\bullet \quad F(s) \qquad \Rightarrow \qquad F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

pa

$$s = \frac{1}{2} \Rightarrow \int_0^\infty \mathrm{e}^{-\frac{x}{2}} x^2 \cos x\, dx = F(\frac{1}{2}) = \cdots = -\frac{176}{125}$$

5. (4 boda)

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0$$

$$f(t) = (u(t) - u(t-1)) - u(t-1) = u(t) - 2u(t-1) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{s} - 2\frac{e^{-s}}{s} = F(s)$$

Preslikavanjem jednadžbe u donje područje dobivamo:

$$s^2 Y(s) - s\cdot y(0) - y'(0) + Y(s) = F(s)$$

$$s^2 Y(s) - s + Y(s) = \frac{1}{s}\left(1 - 2e^{-s}\right)$$

$$Y(s) = \frac{1}{s(s^2+1)} \cdot \left(1 - 2e^{-s}\right) + \frac{s}{s^2+1}$$

$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) \cdot \left(1 - 2e^{-s}\right) + \frac{s}{s^2+1}$$

$$Y(s) = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{2s}{s^2+1}e^{-s} \quad \bullet\!\!-\!\!\circ \quad u(t) - 2u(t-1) + 2\cos(t-1)u(t-1) = y(t)$$

6. (5 boda)

a) (**1b**)

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau = [\text{alternativno}] = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

b) (**2b**) f i g su originali $\Rightarrow$ f i g su eksponencijalnog rasta

$$\begin{aligned} |(f*g)(t)| \quad &\leq \quad \int_0^t |f(\tau)| \cdot |g(t-\tau)|\, d\tau \leq \begin{bmatrix} \exists M_1, M_2 \\ \exists a_1, a_2 \end{bmatrix} \\ &\leq \quad M_1 M_2 \int_0^t e^{a_1 \tau} e^{a_2 (t-\tau)}\, d\tau \end{aligned}$$

$$
\begin{aligned}
&= M_1 M_2 e^{a_2 t} \cdot \int\limits_0^t e^{(a_1-a_2)\tau}\, d\tau \\
&= M_1 M_2 e^{a_2 t} \cdot \frac{1}{a_1-a_2}\left[e^{(a_1-a_2)t}-1\right] \\
&= \frac{M_1 M_2}{a_1-a_2}\left(e^{a_1 t}-e^{a_2 t}\right) = [a = \max\{a_1, a_2\}] \\
&\le \underbrace{\frac{2M_1 M_2}{a_1-a_2}}_{M} e^{at} = M e^{at}
\end{aligned}
$$

c) (**2b**)

$$y(t) = 3\sin t + 2\int\limits_0^t \cos(t-\tau)y(\tau)\, d\tau$$

Slika jednadžbe u donjem području je:

$$Y(s) = 3\cdot\frac{1}{s^2+1} + 2\cdot\frac{s}{s^2+1}\cdot Y(s)$$

$$Y(s)\left(1-\frac{2s}{s^2+1}\right) = \frac{3}{s^2+1}$$

$$Y(s)(s^2-2s+1) = 3$$

$$Y(s) = \frac{3}{(s-1)^2} \quad \bullet\!\!-\!\!\circ \quad 3te^t = y(t)$$

7. (3 boda)

$$e(t) = 1+\cos 2t \quad \circ\!\!-\!\!\bullet \quad \frac{1}{s}+\frac{s}{s^2+4} = E(s)$$

$$Z(s) = \frac{1}{\frac{1}{Ls}+\frac{1}{\frac{1}{Cs}}} = \begin{bmatrix} L=1 \\ C=1 \end{bmatrix} = \cdots = \frac{s}{s^2+1}$$

$$I(s) = \frac{E(s)}{Z(s)} = \frac{s^2+1}{s}\left(\frac{1}{s}+\frac{s}{s^2+4}\right) = \frac{s^2+1}{s^2}+\frac{s^2+1}{s^2+4} = 1+\frac{1}{s^2}+1-\frac{3}{s^2+4}$$

$$I(s) = 2+\frac{1}{s^2}-\frac{3}{2}\cdot\frac{2}{s^2+4} \quad \bullet\!\!-\!\!\circ \quad 2\delta(t)+tu(t)-\frac{3}{2}\sin(2t)u(t) = i(t)$$