Rješenja Završnog ispita iz Matematike 3E
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Pitanja iz 3. ciklusa nastave

1. (3 boda)
$\mathbf{f}(x,y,z)=\mathbf{r}\ln r$, $\mathbf{s}_0=\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$, $T(1,-2,1)$
$\frac{\partial\mathbf{f}}{\partial\mathbf{s}}=\left(\frac{1}{\sqrt{3}}\cdot\frac{\partial}{\partial x}+\frac{1}{\sqrt{3}}\cdot\frac{\partial}{\partial y}+\frac{1}{\sqrt{3}}\cdot\frac{\partial}{\partial z}\right)\mathbf{f}\Big|_T=\cdots=\frac{\ln\sqrt{6}}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

2. (3 boda)
$r=1+\cos\varphi$, $r'=-\sin\varphi$, $x=r\cos\varphi$, $y=r\sin\varphi$, $ds=\sqrt{r^2(\varphi)+(r')^2(\varphi)}$
$\int\limits_\Gamma y\,ds=\int\limits_0^{\frac{\pi}{2}}r(\varphi)\sin\varphi\sqrt{r^2(\varphi)+(r')^2(\varphi)}\,d\varphi=\cdots=\frac{2\sqrt{2}}{5}(4\sqrt{2}-1)$

3. (4 boda)
a) (**1b**) Neka je Γ pozitivno orijentirana zatvorena krivulja, po dijelovima Jordanov luk, te neka je Ω lik u ravnini kojega omeđuje Γ. Nadalje, neka je $\mathbf{f}(x,y)=f_1(x,y)\mathbf{i}+f_2(x,y)\mathbf{j}$ diferencijabilno vektorsko polje. Tada vrijedi:

$$\oint_\Gamma\mathbf{f}\,d\mathbf{r}=\oint_\Gamma f_1\,dx+f_2\,dy=\iint_\Omega\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)dx\,dy.$$

b) (**3b**) $0\le x\le 1$, $x^2\le y\le\sqrt{x}$
$\int\limits_C x^2y\,dx+\left(y+xy^2\right)dy=\iint\limits_D\frac{\partial(y+xy^2)}{\partial x}-\frac{\partial(x^2y)}{\partial y}\,dx\,dy=\iint\limits_D\left(y^2-x^2\right)dx\,dy=$
$\int\limits_0^1dx\int\limits_{x^2}^{\sqrt{x}}\left(y^2-x^2\right)dx\,dy=\cdots=0$

4. (4 boda)
a) (**2b**) Neka polje $\mathbf{f}$ ima neprekinute sve druge parcijalne derivacije na području Ω (tj. $\mathbf{f}$ je klase $\mathbb{C}^2$). Takvo polje je potencijalno onda i samo onda ako je

$$\operatorname{rot}\mathbf{f}=\mathbf{0},$$

i područje Ω je konveksno, tj. svake dvije njegove točke mogu se spojiti segmentom potpuno sadržanim u njemu.

Neka je $\mathbf{f}$ potencijalno polje. Tada postoji skalarno polje $p=p(x,y,z)$ tako da vrijedi $\mathbf{f}=\operatorname{grad}p$. Onda je
$\operatorname{rot}(\mathbf{f})=\operatorname{rot}(\operatorname{grad}p)=\operatorname{rot}\left(\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}+\frac{\partial p}{\partial z}\mathbf{k}\right)=\left(\frac{\partial^2p}{\partial y\partial z}-\frac{\partial^2p}{\partial z\partial y}\right)\mathbf{i}-\left(\frac{\partial^2p}{\partial x\partial z}-\frac{\partial^2p}{\partial z\partial x}\right)\mathbf{j}+$
$\left(\frac{\partial^2p}{\partial x\partial y}-\frac{\partial^2p}{\partial y\partial x}\right)\mathbf{k}=\mathbf{0}$
b) (**2b**) $\operatorname{rot}\mathbf{a}=\cdots=(2z-2z)\mathbf{i}-(0-0)\mathbf{j}+(0-0)\mathbf{k}=\mathbf{0}$ pa je $\mathbf{a}$ potencijalno polje.
$p(x,y,z)=\int\limits_{x_0}^x\frac{1}{t^3}\,dt+\int\limits_{y_0}^y z^2\,ds+\int\limits_{z_0}^z 2uy_0\,du+C=\cdots=-\frac{1}{2x^2}+z^2y+K$

5. (3 boda)
$z = 2(x^2+y^2),\ 0 \le z \le 2,\ z = z(x,y) \Rightarrow$ uzimamo projekciju Ω_{xy},
$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+16x^2+16y^2}$
Ω_{xy} je krug sa središtem u $(0,0)$ radijusa $r=1$. U polarnim koordinatama: $x = r\cos\varphi$, $y = r\sin\varphi$, $dxdy = rdrd\varphi$.
$\iint\limits_S dS = \iint\limits_{\Omega_{xy}} \sqrt{1+16x^2+16y^2}\,dx\,dy = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 r\sqrt{1+16r^2}\,dr = \cdots = \frac{\pi}{24}\left(\sqrt{17^3}-1\right)$

6. (3 boda)
$y = x^2$, $0 \le z \le 4-y = 4-x^2$
$\iint\limits_S z\,dx\,dy - 3\,dx\,dz = \underbrace{\iint\limits_{\Omega_{xy}} z\,dx\,dy}_{0} - 3\iint\limits_{\Omega_{xz}} dx\,dz = -3\int\limits_{-2}^{2} dx \int\limits_0^{4-x^2} dz = \cdots = -32$

7. (5 bodova)
a) (**2b**) Neka je V zatvoreno područje u prostoru omeđeno sa po dijelovima glatkom zatvorenom plohom S koja samu sebe ne presijeca i neka je $\mathbf{n}$ polje jediničnih vanjskih normala na S. Ako je $\mathbf{a}$ vektorsko polje klase $\mathbb{C}^1$ u okolini područja V, onda vrijedi formula

$$\iiint_V (\operatorname{div}\mathbf{a})\,dx\,dy\,dz = \iint_S \mathbf{a}\cdot\mathbf{n}\,dS.$$

b) (**3b**) $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
$I = \iint\limits_{S^+} x\,dy\,dz + y\,dx\,dz + z\,dx\,dy = \iint\limits_{S^+} \mathbf{a}\cdot d\mathbf{S} = \int\int\limits_{S^+\cup\sigma_1\cup\sigma_2} \mathbf{a}\cdot d\mathbf{S} -$
$- \iint\limits_{\sigma_1} \mathbf{a}\cdot d\mathbf{S} - \iint\limits_{\sigma_2} \mathbf{a}\cdot d\mathbf{S}$
$\int\int\limits_{S^+\cup\sigma_1\cup\sigma_2} \mathbf{a}\cdot d\mathbf{S} = \iiint\limits_V (\operatorname{div}\mathbf{a})\,dV = 3\iiint\limits_V dV = 3\cdot 1^2\cdot\pi\cdot 1 = 3\pi$
$\iint\limits_{\sigma_1} \mathbf{a}\cdot d\mathbf{S} = \iint\limits_{\sigma_1} z\,dx\,dy = \iint\limits_{\Omega_{xy}} 2\,dx\,dy = 2\cdot 1^2\cdot\pi = 2\pi$
$\iint\limits_{\sigma_2} \mathbf{a}\cdot d\mathbf{S} = \iint\limits_{\sigma_2} z\,dx\,dy = -\iint\limits_{\Omega_{xy}} 1\,dx\,dy = -1\cdot 1^2\cdot\pi = -\pi$
$I = 3\pi - 2\pi - (-\pi) = 2\pi$

Pitanja iz cijelog gradiva

8. (4 boda)
Prelazimo na pomaknute sferne koordinate:

$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi + 1,\ z = r\cos\theta,\ dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$

$\iiint\limits_V \left(x^2+z^2\right)\,dx\,dy\,dz \;=\; \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\theta \int\limits_0^{2} r^2\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) r^2\sin\theta\,dr \;=$

$$\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \left(\sin^3\theta\cos^2\varphi + \cos^2\theta\sin\theta\right)}_{I_1+I_2} \cdot \underbrace{\left(\frac{r^5}{5}\bigg|_0^2\right)}_{\frac{32}{5}} d\theta$$

$I_1 = \int\limits_0^{2\pi} \cos^2\varphi\,d\varphi \int\limits_0^{\pi} \sin^3\theta\,d\theta = \cdots = \frac{4}{3}\pi$

$I_2 = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} \cos^2\theta\sin\theta\,d\theta = \cdots = \frac{4}{3}\pi$

$I = (I_1+I_2)\cdot\frac{32}{5} = \frac{32}{5}\left(\frac{4}{3}\pi + \frac{4}{3}\pi\right) = \frac{256}{15}\pi$

9. (3 boda)

$f(x) = x$ je neparna funkcija $\Rightarrow a_n = 0 \quad \forall n$

interval $[-1,1] \Rightarrow L = 1$

$f(x) = \sum\limits_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$b_n = \frac{2}{L}\int\limits_0^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 2\int\limits_0^{1} x\sin n\pi x\,dx = \cdots = 2\frac{(-1)^{n+1}}{n\pi}$

$f(x) = \sum\limits_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n\pi}\sin n\pi x$

10. (3 boda)

$y' - 2y = f(t), \quad y(0) = 1$

$f(t) = u(t) - u(t-1)$

$sY(s) - 1 - 2Y(s) = \frac{1}{s} - \frac{\mathrm{e}^{-s}}{s}$

$\vdots$

$Y(s) = \frac{3}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{\mathrm{e}^{-s}}{s-2} + \frac{1}{2}\frac{\mathrm{e}^{-s}}{s}$

$y(t) = u(t)\left(\frac{3}{2}\mathrm{e}^{2t} - \frac{1}{2}\right) + u(t-1)\left(\frac{1}{2} - \frac{1}{2}\mathrm{e}^{2(t-1)}\right)$