

FORMULE VEZANE UZ MATEMATIČKE KOLEGIJE PREDDIPLOMSKOG STUDIJA

Vrijednosti sinusa i kosinusa					
φ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \varphi$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \varphi$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Trigonometrijske funkcije polovičnih argumenata	
$\sin^2 \frac{x}{2}$	$= \frac{1 - \cos x}{2}$
$\cos^2 \frac{x}{2}$	$= \frac{1 + \cos x}{2}$

Trigonometrijske funkcije višestrukih argumenata	
$\sin 2x$	$= 2 \sin x \cos x$
$\cos 2x$	$= \cos^2 x - \sin^2 x$
$\operatorname{tg} 2x$	$= \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}$
$\operatorname{ctg} 2x$	$= \frac{\operatorname{ctg} x - 1}{2 \operatorname{ctg} x}$

Veza s tangensom	
$\sin^2 x$	$= \frac{\operatorname{tg}^2 x}{1 + \operatorname{tg}^2 x}$
$\cos^2 x$	$= \frac{1}{1 + \operatorname{tg}^2 x}$
$\sin x$	$= \frac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}}$
$\cos x$	$= \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}}$

Adicijski teoremi	
$\sin(x \pm y)$	$= \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y)$	$= \cos x \cos y \mp \sin x \sin y$
$\operatorname{tg}(x \pm y)$	$= \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}$
$\operatorname{ctg}(x \pm y)$	$= \frac{\operatorname{ctg} x \operatorname{ctg} y \mp 1}{\operatorname{ctg} y \pm \operatorname{ctg} x}$

Formule pretvorbe	
$\sin x \cos y$	$= \frac{1}{2}(\sin(x + y) + \sin(x - y))$
$\cos x \cos y$	$= \frac{1}{2}(\cos(x + y) + \cos(x - y))$
$\sin x \sin y$	$= \frac{1}{2}(\cos(x - y) - \cos(x + y))$
$\sin x + \sin y$	$= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
$\sin x - \sin y$	$= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$
$\cos x + \cos y$	$= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$
$\cos x - \cos y$	$= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

Tablica derivacija	
$f(x)$	$f'(x)$
x^a	ax^{a-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$
$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$
$\operatorname{sh} x$	$\operatorname{ch} x$
$\operatorname{ch} x$	$\operatorname{sh} x$
$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$
$\operatorname{arsh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arch} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arth} x$	$\frac{1}{1-x^2}$
$\operatorname{arcth} x$	$\frac{1}{1-x^2}$

Neodređeni integrali
1) $\int \frac{dx}{x} = \ln x + C$
2) $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \in \mathbf{R} \setminus \{-1\}$
3) $\int a^x dx = \frac{a^x}{\ln a} + C$
4) $\int e^x dx = e^x + C$
5) $\int \sin x dx = -\cos x + C$
6) $\int \cos x dx = \sin x + C$
7) $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$
8) $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$
9) $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{arctg} \left(\frac{x}{a} \right) + C, a > 0$
10) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right + C, a > 0$
11) $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \left(\frac{x}{a} \right) + C, a > 0$
12) $\int \frac{dx}{\sqrt{x^2+A}} = \ln x + \sqrt{x^2+A} + C, A \neq 0$
13) $\int \operatorname{sh} x dx = \operatorname{ch} x + C$
14) $\int \operatorname{ch} x dx = \operatorname{sh} x + C$
15) $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C$
16) $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$

Projekcija vektora na vektor

$$\mathbf{b}_a = \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} \cos \angle(\mathbf{a}, \mathbf{b}) \cdot \mathbf{a}$$

Prikaz vektora d u ortogonalnoj bazi (a, b, c)

$$\mathbf{d} = \frac{\mathbf{d} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} + \frac{\mathbf{d} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} + \frac{\mathbf{d} \cdot \mathbf{c}}{\|\mathbf{c}\|^2} \mathbf{c}$$

Udaljenost točke od ravnine

$$d(T_0, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Kut između dviju ravnina

$$\cos \angle(\pi_1, \pi_2) = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|},$$

gdje su \mathbf{n}_1 i \mathbf{n}_2 odgovarajući vektori normale ravnina π_1 i π_2

Udaljenost točke od pravca u prostoru

$$d(T_1, p) = \frac{\|(\mathbf{r}_{T_1} - \mathbf{r}_{T_0}) \times \mathbf{c}\|}{\|\mathbf{c}\|},$$

gdje su \mathbf{r}_{T_0} i \mathbf{r}_{T_1} radij vektori točaka T_0 i T_1 ,
a \mathbf{c} vektor smjera pravca koji prolazi točkom T_0

Kut između pravca i ravnine

$$\cos(90^\circ - \angle(p, \pi)) = \frac{|\mathbf{c} \cdot \mathbf{n}|}{\|\mathbf{c}\| \|\mathbf{n}\|},$$

gdje je \mathbf{c} vektor smjera pravca p , a \mathbf{n} odgovarajući vektori normale ravnine π

Trigonometrijski Fourierov red

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T} \right)$$
$$a_0 = \frac{2}{T} \int_a^b f(x) dx$$
$$a_n = \frac{2}{T} \int_a^b f(x) \cos \frac{2n\pi x}{T} dx$$
$$b_n = \frac{2}{T} \int_a^b f(x) \sin \frac{2n\pi x}{T} dx$$

Parsevalova jednakost

$$\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 = \frac{2}{T} \int_a^b |f(x)|^2 dx$$

Fourierov integral

$$f(x) \sim \frac{1}{\pi} \int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi) \cos \lambda(x - \xi) d\xi$$

Fourierov integral i spektar

$$f(x) \sim \int_0^{\infty} (A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x) d\lambda$$
$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \cos \lambda \xi d\xi$$
$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \sin \lambda \xi d\xi$$

Fourierova transformacija

$$\hat{f}(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda\xi} f(\xi) d\xi$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{f}(\lambda) d\lambda$$

Laplaceova transformacija

$$F(s) = \mathcal{L}(f(t)) := \int_0^{\infty} e^{-st} f(t) dt$$

Svojstva Laplaceovih transformata	
Linearnost	$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha F(s) + \beta G(s)$
Teorem prigušenja	$\mathcal{L}(e^{-at} f(t)) = F(s + a)$
Teorem o pomaku	$\mathcal{L}(f(t - a)u(t - a)) = e^{-as} F(s)$
Tm o deriviranju originala	$\mathcal{L}(f'(t)) = sF(s) - f(0)$
Tm o deriviranju slike	$\mathcal{L}((-t)f(t)) = F'(s)$
Tm o integriranju slike	$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s) ds$
Tm o integriranju originala	$\mathcal{L}\left(\int_0^t f(\tau) d\tau\right) = \frac{F(s)}{s}$
Slika periodične funkcije	$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$
Teorem o konvoluciji	$(f_1 * f_2)(t) := \int_0^t f_1(\tau) f_2(t - \tau) d\tau$ $\mathcal{L}(f_1 * f_2) = F_1(s) F_2(s)$

Tablica Laplaceovih transformata	
$F(s)$	$f(t)$
s^n	$\delta^{(n)}(t)$
s	$\delta'(t)$
1	$\delta(t)$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s^n}, n \in \mathbf{N}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{n!}{s^{n+1}}, n \in \mathbf{N}$	t^n
$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{(s-a)^2}$	te^{at}
$\frac{1}{(s-a)^n}, n \in \mathbf{N}$	$\frac{t^{n-1}e^{at}}{(n-1)!}$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{s^2+a^2}$	$\frac{1}{a} \sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a} \operatorname{sh} at$
$\frac{s}{s^2-a^2}$	$\operatorname{ch} at$
$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3} (\sin at - at \cos at)$
$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a} \sin at$

Zamjena varijabli u trostrukom integralu.

Neka su $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$

neprekidno diferencijabilne bijekcije između omeđenih područja V' , $V \subseteq \mathbb{R}^3$.
Tada vrijedi

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{V'} f(x(u, v, w), y(u, v, w), z(u, v, w)) |J| du dv dw,$$

pri čemu je J Jakobijan definiran izrazom

$$J = \begin{vmatrix} \partial_u x & \partial_v x & \partial_w x \\ \partial_u y & \partial_v y & \partial_w y \\ \partial_u z & \partial_v z & \partial_w z \end{vmatrix}$$

Koordinatni sustavi u \mathbb{R}^2 i \mathbb{R}^3 .

- Polarne koordinate

$$\begin{aligned} x &= r \cos \varphi, \\ y &= r \sin \varphi, \quad r \in [0, +\infty), \varphi \in [0, 2\pi) \end{aligned}$$

Jakobijan: $J = r$.

- Cilindrične koordinate

$$\begin{aligned} x &= r \cos \varphi, \\ y &= r \sin \varphi, \\ z &= z, \quad r \in [0, +\infty), \varphi \in [0, 2\pi), z \in (-\infty, +\infty) \end{aligned}$$

Jakobijan: $J = r$.

- Sferne koordinate

$$\begin{aligned} x &= r \sin \theta \cos \varphi, \\ y &= r \sin \theta \sin \varphi, \\ z &= r \cos \theta, \quad r \in [0, +\infty), \varphi \in [0, 2\pi), \theta \in [0, \pi]. \end{aligned}$$

Jakobijan: $J = r^2 \sin \theta$.

Neke relacije iz vektorske algebre.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}),$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}),$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

Vektorska analiza.

- usmjerena derivacija skalarnog polja f

$$\frac{\partial f}{\partial \mathbf{s}} = (\mathbf{s}_0 \cdot \nabla)f = \mathbf{s}_0 \cdot \nabla f = s_{01} \frac{\partial f}{\partial x} + s_{02} \frac{\partial f}{\partial y} + s_{03} \frac{\partial f}{\partial z}$$

- usmjerena derivacija vektorskog polja \mathbf{f}

$$\frac{\partial \mathbf{f}}{\partial \mathbf{s}} = (\mathbf{s}_0 \cdot \nabla)\mathbf{f} = (\mathbf{s}_0 \cdot \nabla)f_1 \mathbf{i} + (\mathbf{s}_0 \cdot \nabla)f_2 \mathbf{j} + (\mathbf{s}_0 \cdot \nabla)f_3 \mathbf{k}$$

pri čemu je $\mathbf{s}_0 = \frac{\mathbf{s}}{|\mathbf{s}|}$, te $\nabla = \mathbf{i} \partial_x + \mathbf{j} \partial_y + \mathbf{k} \partial_z$

i $\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$.

- Laplaceov operator

- $\Delta f = \text{div grad } f$, f skalarno polje
- $\Delta \mathbf{f} = \text{grad div } \mathbf{f} - \text{rot rot } \mathbf{f}$, f vektorsko polje

Računanje potencijala vektorskog polja. Neka je vektorsko polje $\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ potencijalno, odnosno vrijedi $\nabla p = \mathbf{f}$. Tada je

$$p(x, y, z) = \int_{x_0}^x f_1(t, y, z) dt + \int_{y_0}^y f_2(x_0, t, z) dt + \int_{z_0}^z f_3(x_0, y_0, t) dt + C,$$

gdje je $C \in \mathbb{R}$.

Greenova formula. Neka je Ω 1-povezano područje u \mathbb{R}^2 , Γ pozitivno orijentirana zatvorena krivulja, koja omeđuje Ω , te neka je $\mathbf{f}(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}$ diferencijabilno vektorsko polje.

Tada vrijedi:

$$\oint_{\Gamma} \mathbf{f} \cdot d\mathbf{r} = \oint_{\Gamma} f_1 dx + f_2 dy = \iint_{\Omega} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy.$$

Teorem o divergenciji. Neka je S po dijelovima glatka, zatvorena ploha koja omeđuje volumen V . Nadalje, neka je \mathbf{n} polje jediničnih vanjskih normala na S , te \mathbf{f} vektorsko polje klase C^1 u okolini područja V . Tada je

$$\iint_S \mathbf{f} \cdot \mathbf{n} dS = \iiint_V \text{div } \mathbf{f} dV,$$

odnosno, u skalarnom zapisu

$$\iint_S f_1 dy dz + f_2 dx dz + f_3 dx dy = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx dy dz,$$

gdje je $\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$.

Stokesova formula. Neka je \mathbf{f} vektorsko polje klase C^1 na području koje sadrži plohu S sa glatkim rubom Γ , te neka su ploha S i krivulja Γ koherentno orjentirane. Tada je

$$\oint_{\Gamma} \mathbf{f} \cdot d\mathbf{r} = \iint_S \operatorname{rot} \mathbf{f} \cdot \mathbf{n} dS,$$

odnosno, u skalarnom zapisu

$$\begin{aligned} \oint_{\Gamma} f_1 dx + f_2 dy + f_3 dz &= \iint_S \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) dydz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) dzdx \\ &\quad + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy, \end{aligned}$$

gdje je $\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$.

Uvjetna vjerojatnost

$P(A B) = \frac{P(AB)}{P(B)}, \quad \text{gdje je } P(B) > 0$

Formula potpune vjerojatnosti

$P(A) = \sum_{i=1}^n P(H_i) P(A H_i),$
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gdje je $\{H_1, H_2, \dots, H_n\}$ -potpun sustav događaja
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Bayesova formula

$P(H_j A) = \frac{P(H_j) P(A H_j)}{\sum_{i=1}^n P(H_i) P(A H_i)},$
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gdje je $\{H_1, H_2, \dots, H_n\}$ -potpun sustav događaja
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Binomna razdioba $X \sim \mathcal{B}(n, p)$
$X \in \{0, 1, 2, \dots, n\}$, $p_k = P(X = k) = \binom{n}{k} p^k q^{n-k}$, gdje je $q = 1 - p$
$E(X) = np$, $D(X) = npq$
$\mathcal{B}(n, p) \approx \mathcal{N}(np, npq)$ za dovoljno veliki n
$\mathcal{B}(n, p) \approx \mathcal{P}(np)$ za dovoljno veliki n i dovoljno malen p

Poissonova razdioba $X \sim \mathcal{P}(\lambda)$
$X \in \{0, 1, 2, \dots\}$, $p_k = P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$, gdje je $\lambda > 0$
$E(X) = \lambda$, $D(X) = \lambda$

Eksponencijalna razdioba $X \sim \mathcal{E}(\lambda)$
$X \in \mathbb{R}^+$, $f(x) = \lambda e^{-\lambda x}$, $x > 0$, $F(x) = 1 - e^{-\lambda x}$, $x > 0$, gdje je $\lambda > 0$
$E(X) = \frac{1}{\lambda}$, $D(X) = \frac{1}{\lambda^2}$

Normalna razdioba $X \sim \mathcal{N}(a, \sigma^2)$
$X \in \mathbb{R}$, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$, gdje je $a \in \mathbb{R}$ i $\sigma^2 \in \mathbb{R}^+$
$E(X) = a$, $D(X) = \sigma^2$
$X \sim \mathcal{N}(a, \sigma^2) \implies \frac{X - a}{\sigma} \sim \mathcal{N}(0, 1)$