Metoda konačnih elemenata (MKE) u elektromagnetizmu

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Pregled predavanja

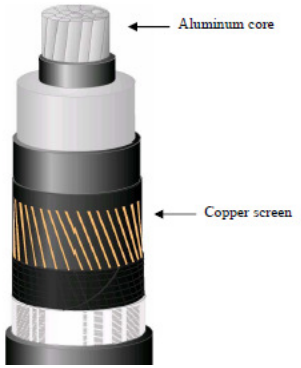
- Numeričke metode u elektromagnetizmu
- Osnovne osobine metode konačnih elemenata
- Primjer razvoja metode konačnih elemenata na elektrostatskom 1D modelu
- Primjer primjene MKE u programskim paketima

NUMERIČKE METODE U ELEKTROMAGNETIZMU

- Proračun elektromagnetskog polja važan je dio proračuna električnih uređaja (radne karakteristike uređaja, standardi, EMC)
- Metode proračuna
  - Najčešće metoda konačnih elemenata (MKE, eng. Finite element methods, FEM)
    - Programski paketi Ansys, Infolytica MagNet, ElecNet, VectorFields Opera, CEDRAT Flux
  - Metoda rubnih elemenata (Boundary element Methods, BEM)
    - Programski paketi IntegratedSoft
  - Metoda konačnih razlika u vremenskoj domeni (FDTD, Finite difference Time Domain methods)

METODA KONAČNIH ELEMENATA

- Danas je praktički standard u industrijskim proračunima u elektromagnetizmu
- Višedisciplinarno područje
  - Modeliranje geometrije problema
  - Matematičko modeliranje
  - Matrična algebra
  - Numeričke metode u proračunu
  - Programiranje
- Dodatno – problemi različitih proračuna u međudjelovanju
  - Elektromagnetski, termički, dinamika fluida



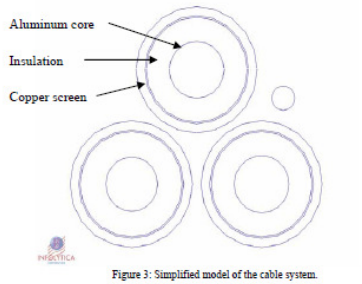
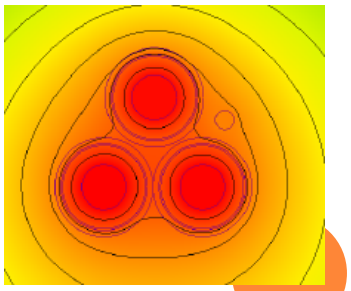
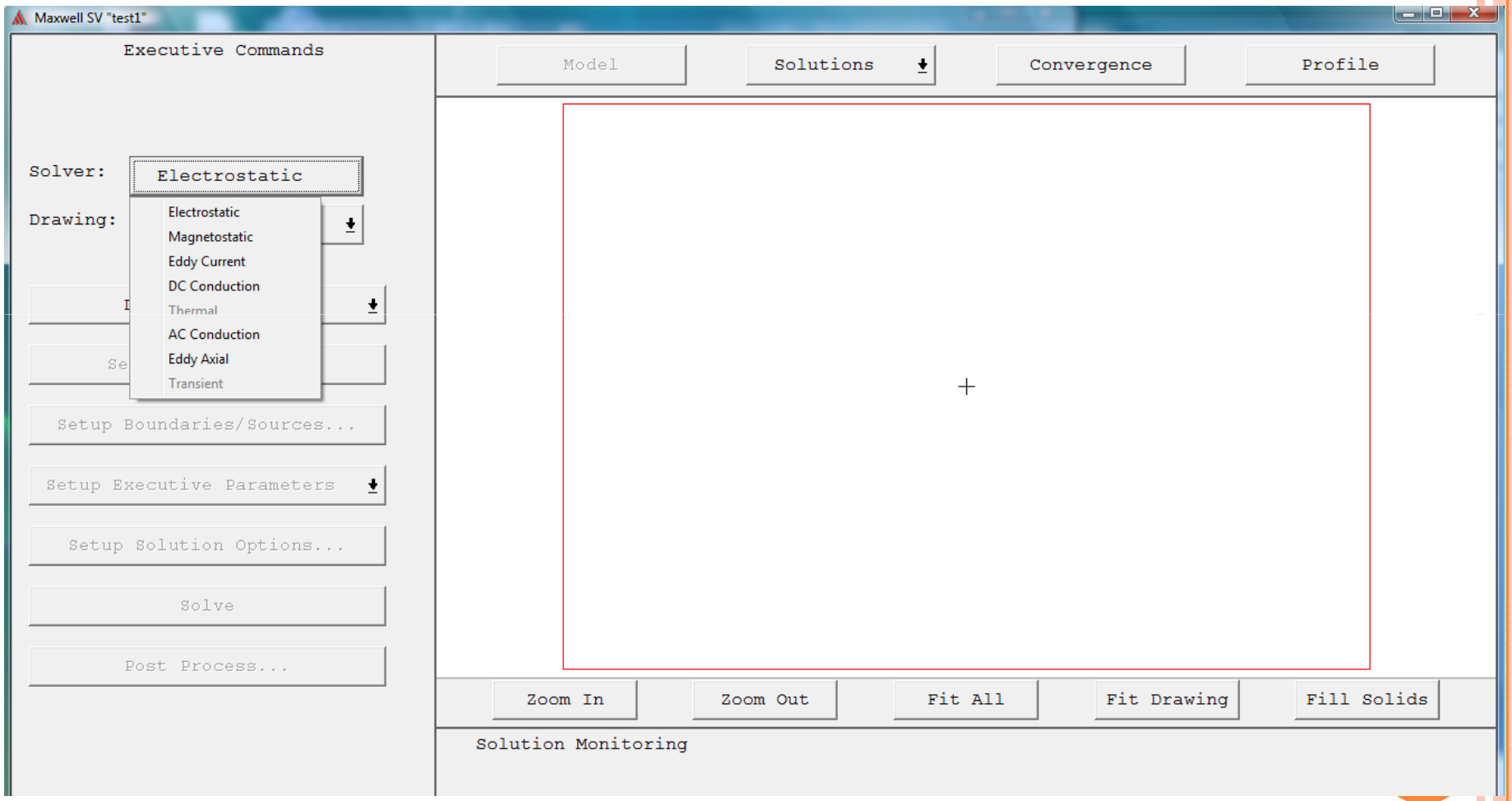


Figure 3: Simplified model of the cable system.



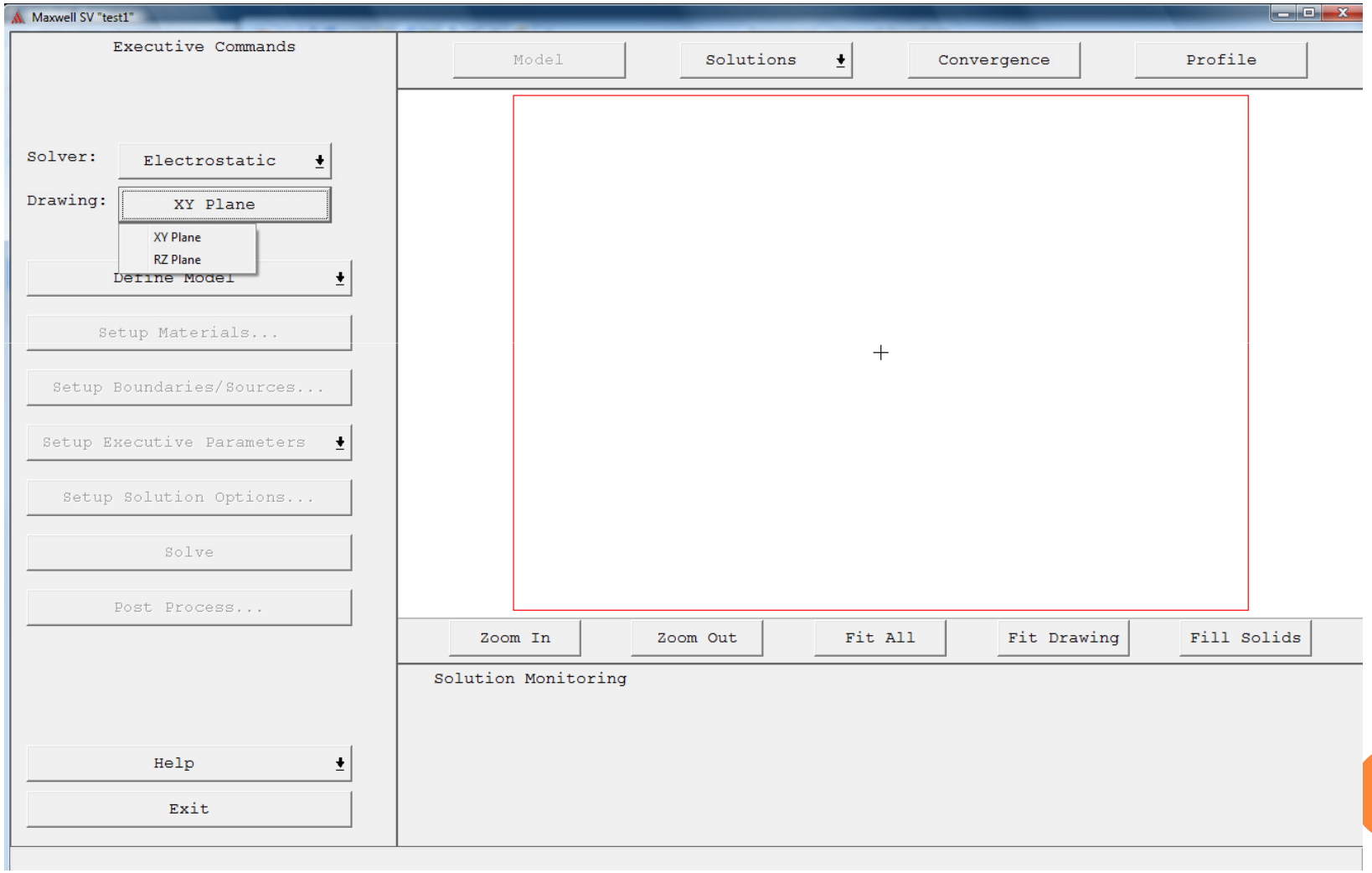
METODA KONAČNIH ELEMENATA – PRISTUP RJEŠAVANJU PRIMJENOM PROGRAMSKOG PAKETA



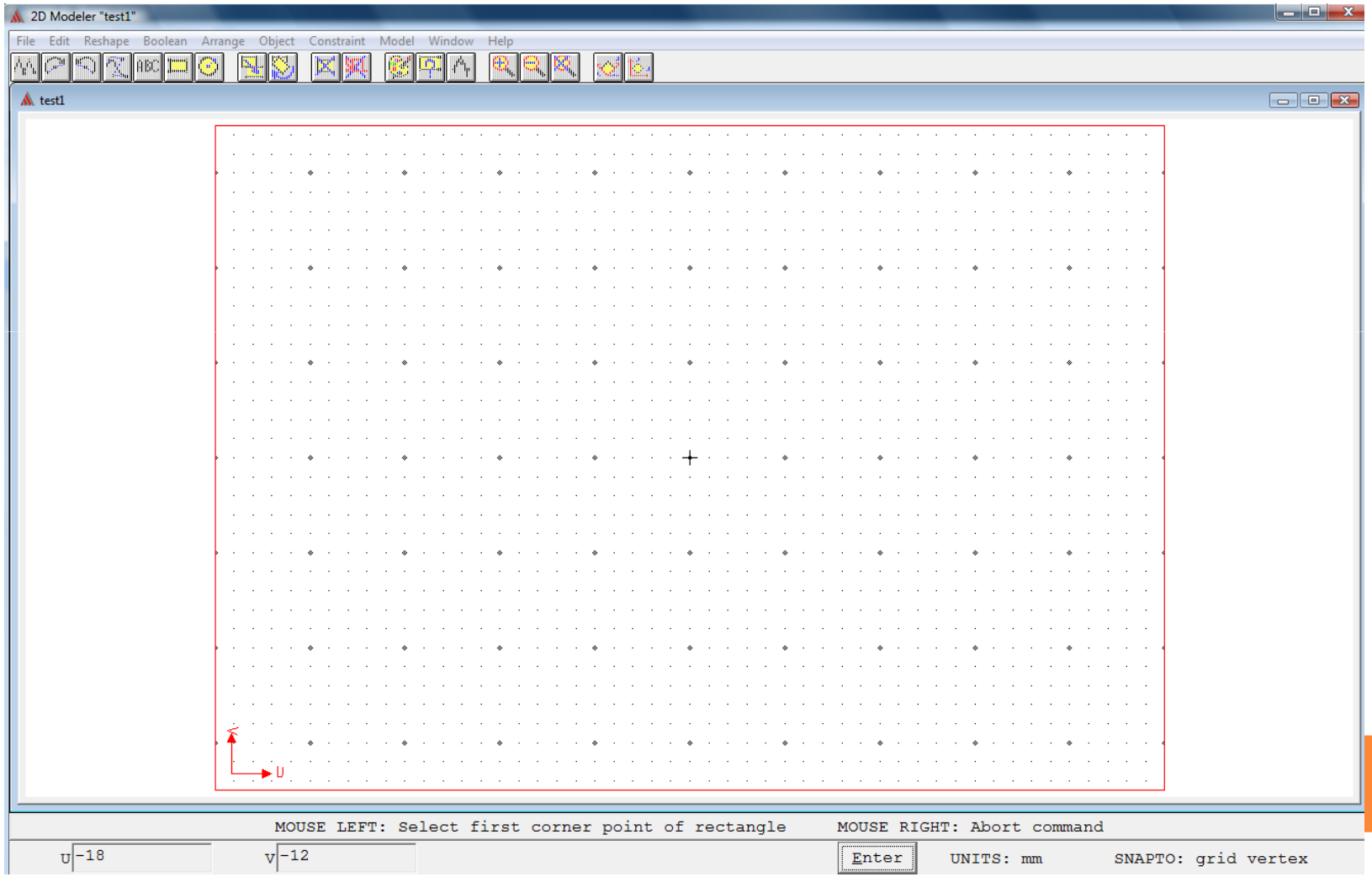
MAXWELL – DEFINICIJA MODELA PRORAČUNA

- Proračun električnog polja
  - Elektrostatsko polje – uvjetovano statičkim naponom na vodičima i razdiobom statičkih naboja na vodiču
  - DC vođenje – analiza prilika u vodičima i dielektricima uslijed statičkog polja
  - AC vođenje – analiza prilika u vodičima i dielektricima uslijed vremenski promjenjivog polja

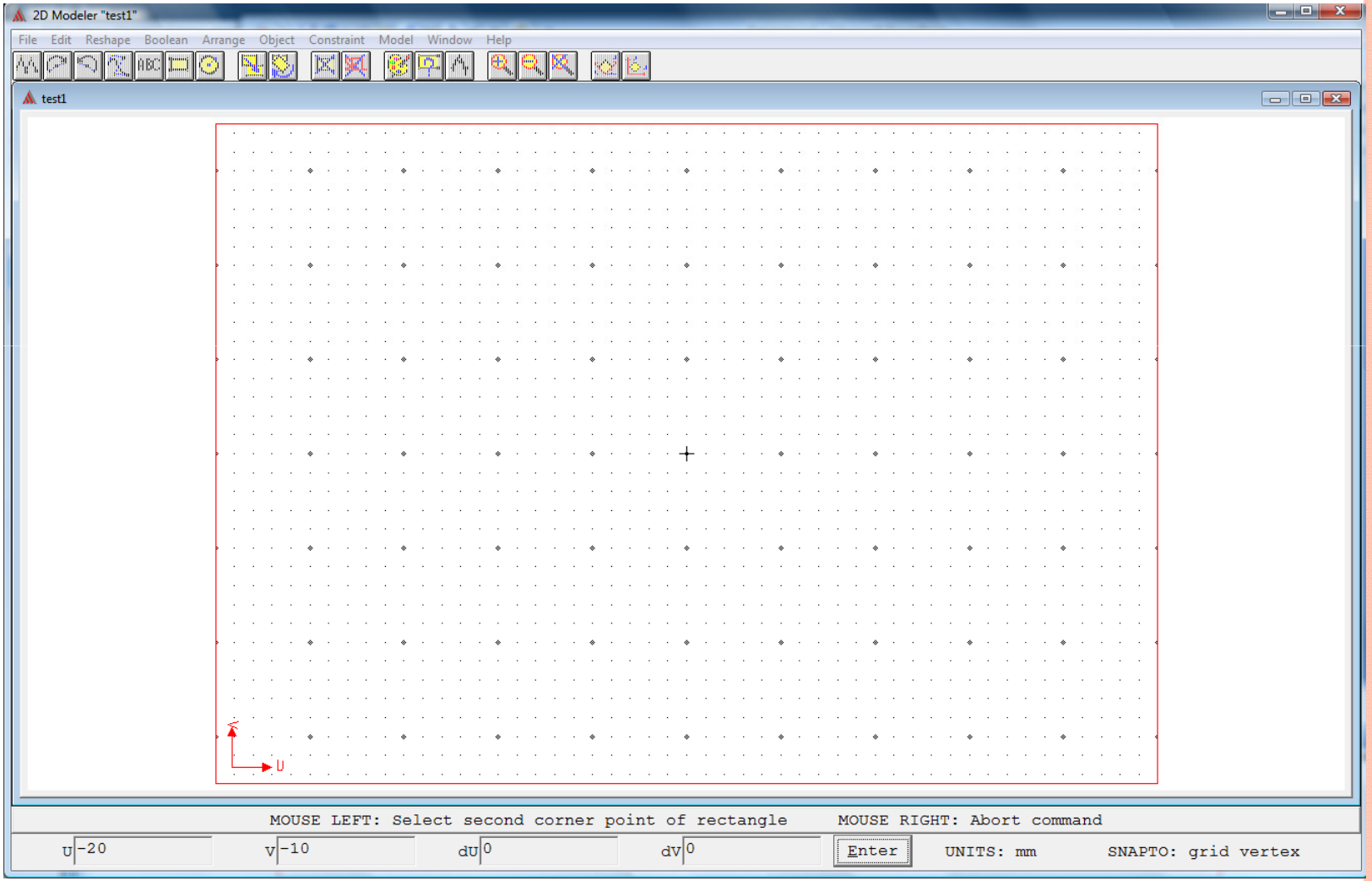
MAXWELL – DEFINICIJA GEOMETRIJE U 2D



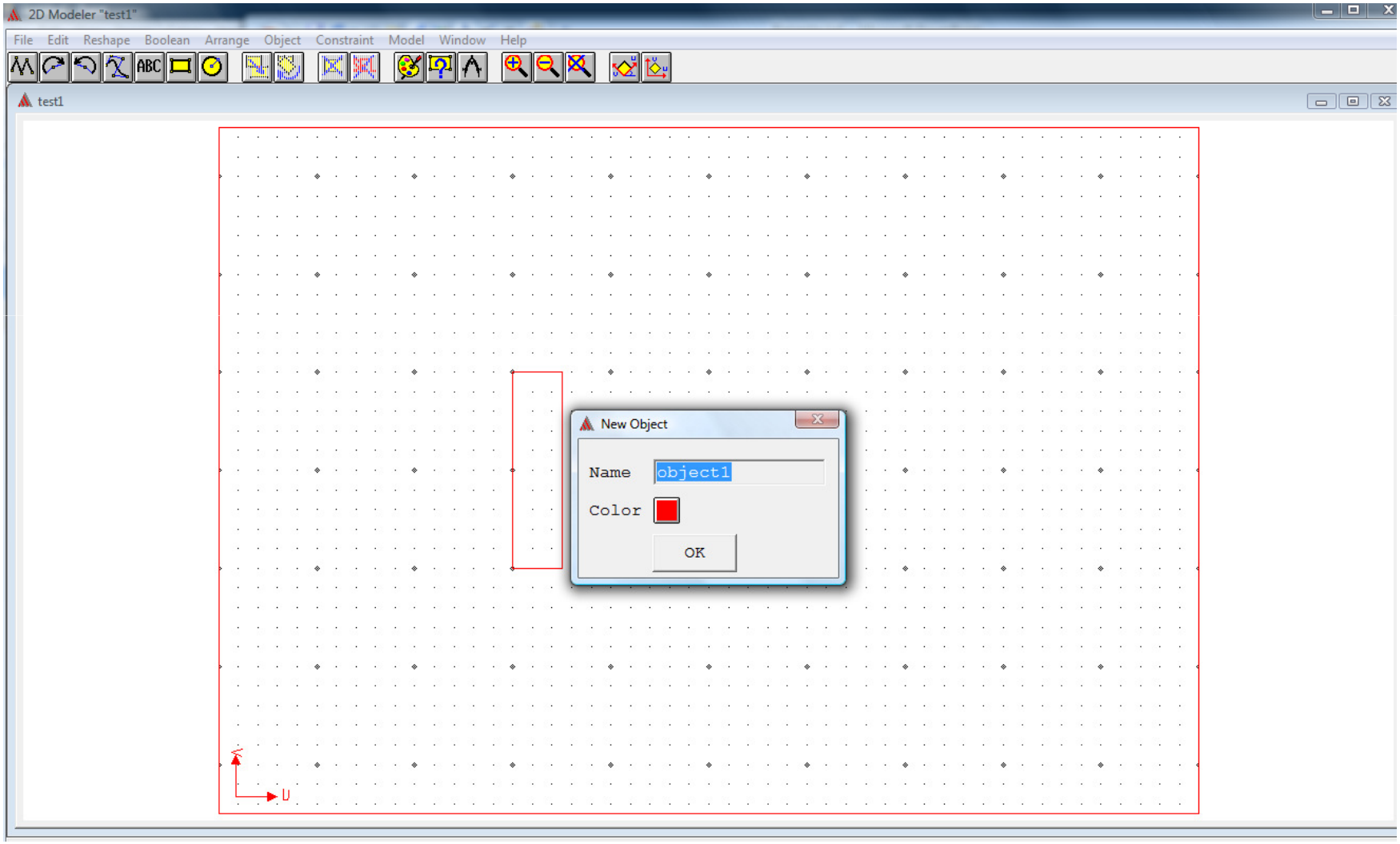
Maxwell – Modeliranje geometrije



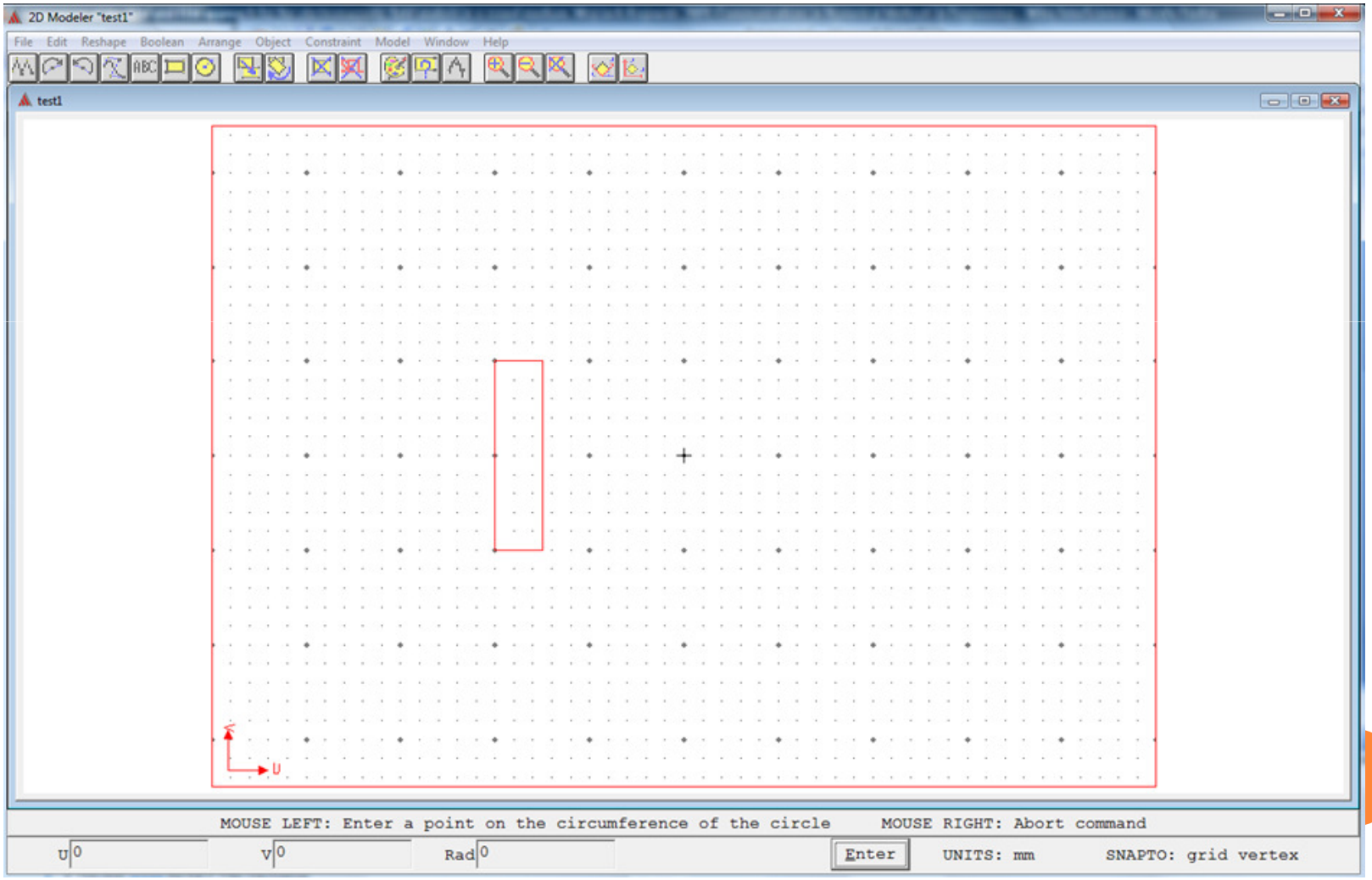
MAXWELL – KREIRANJE OBJEKATA 1/5



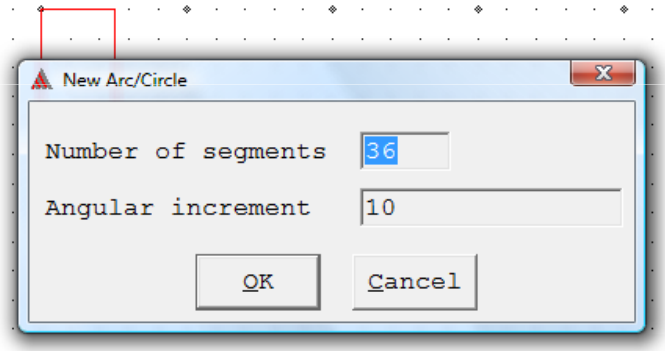
MAXWELL – KREIRANJE OBJEKATA 2/5



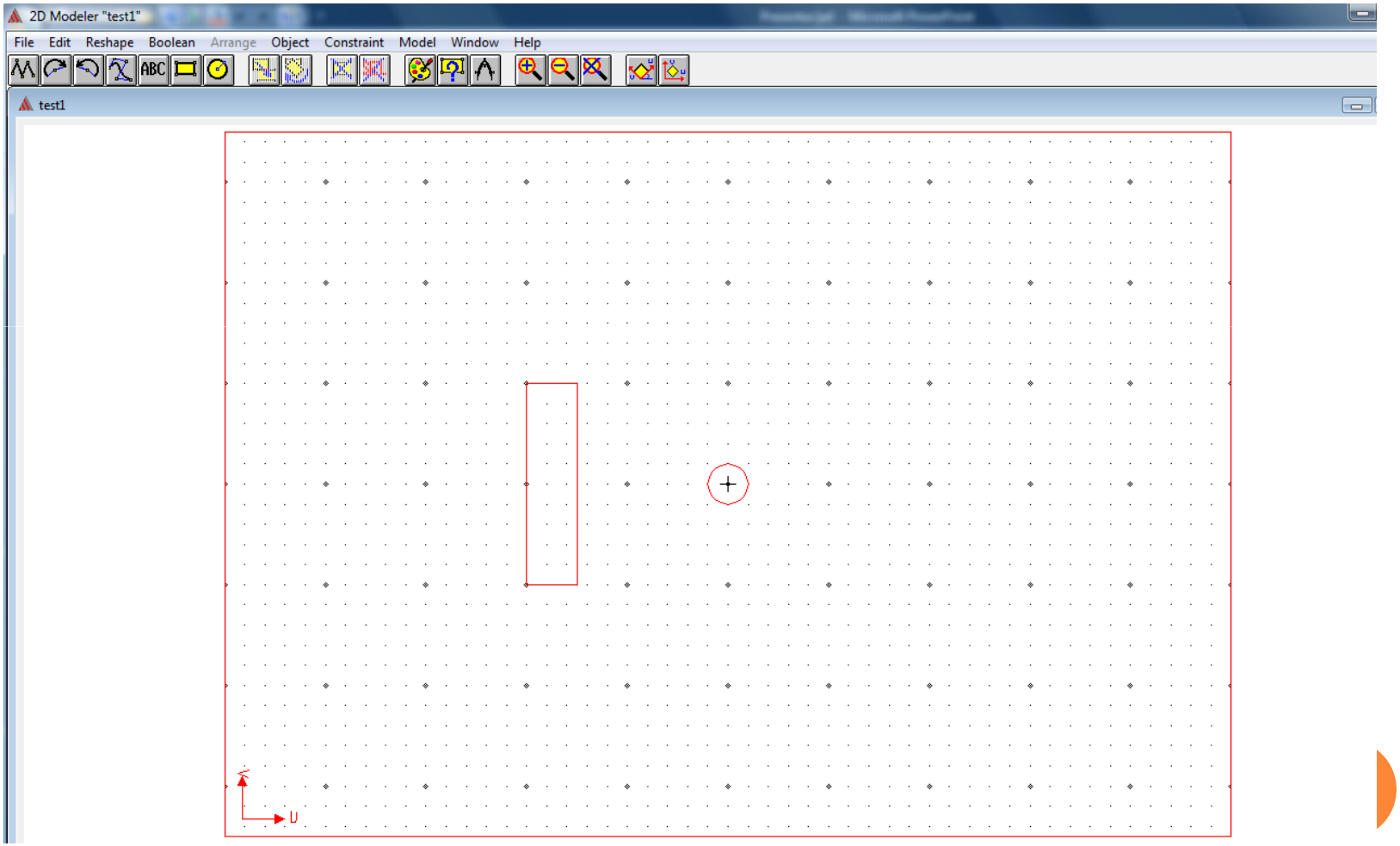
Maxwell – Kreiranje objekata 3/5



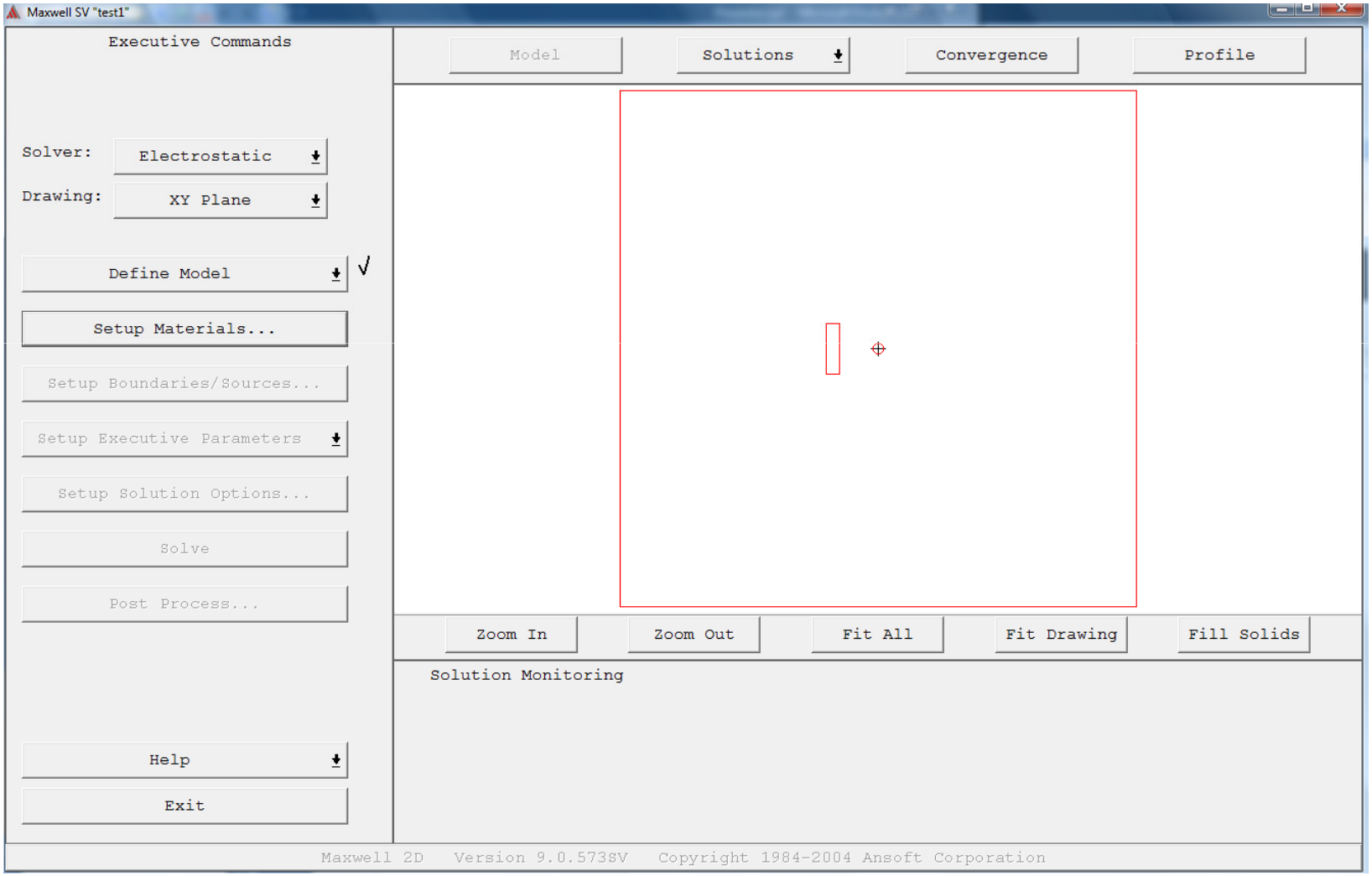
Maxwell – Kreiranje objekata 4/5



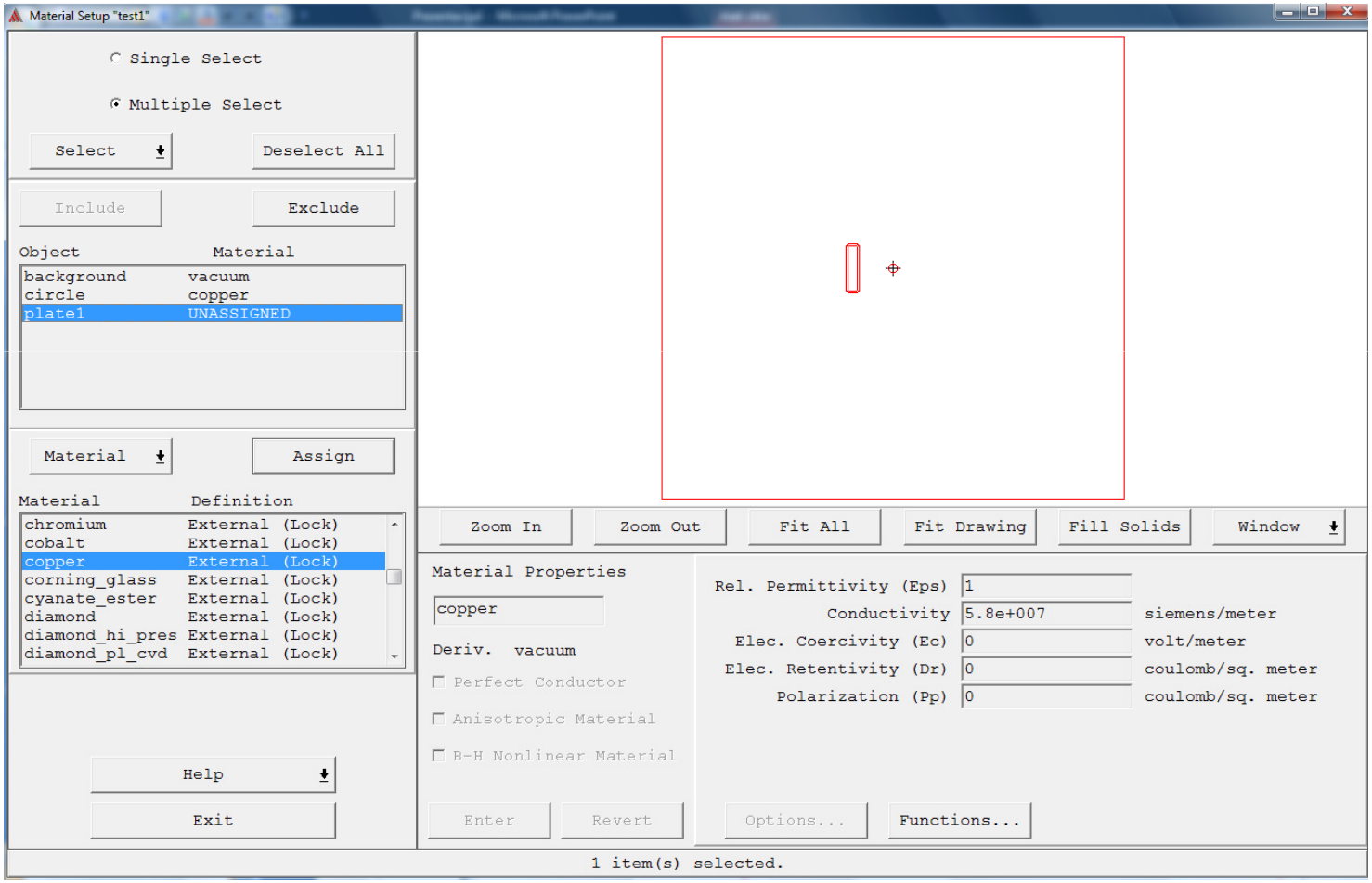
MAXWELL – KREIRANJE OBJEKATA 5/5



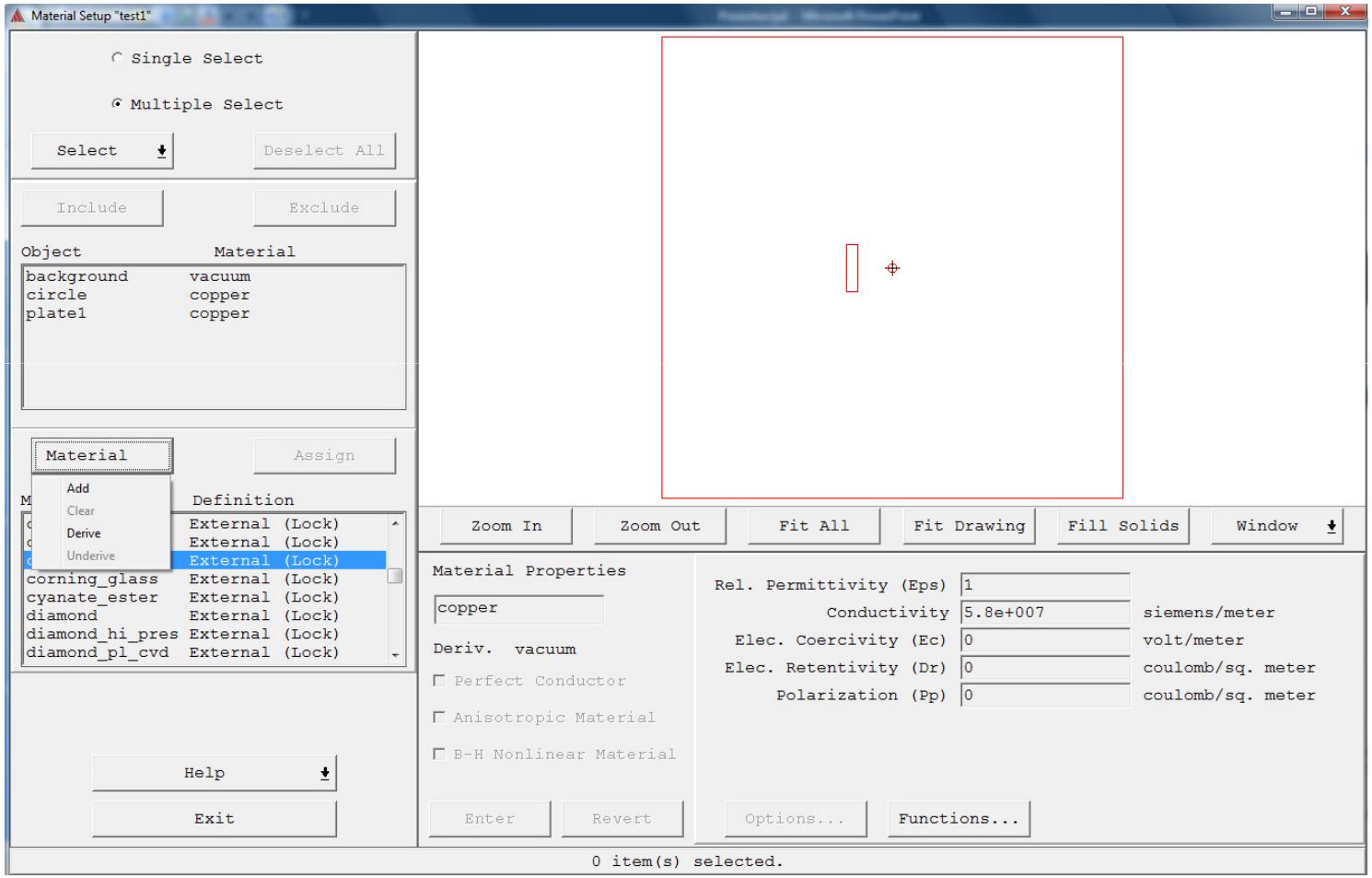
MAXWELL – DEFINICIJA MATERIJALA1/3



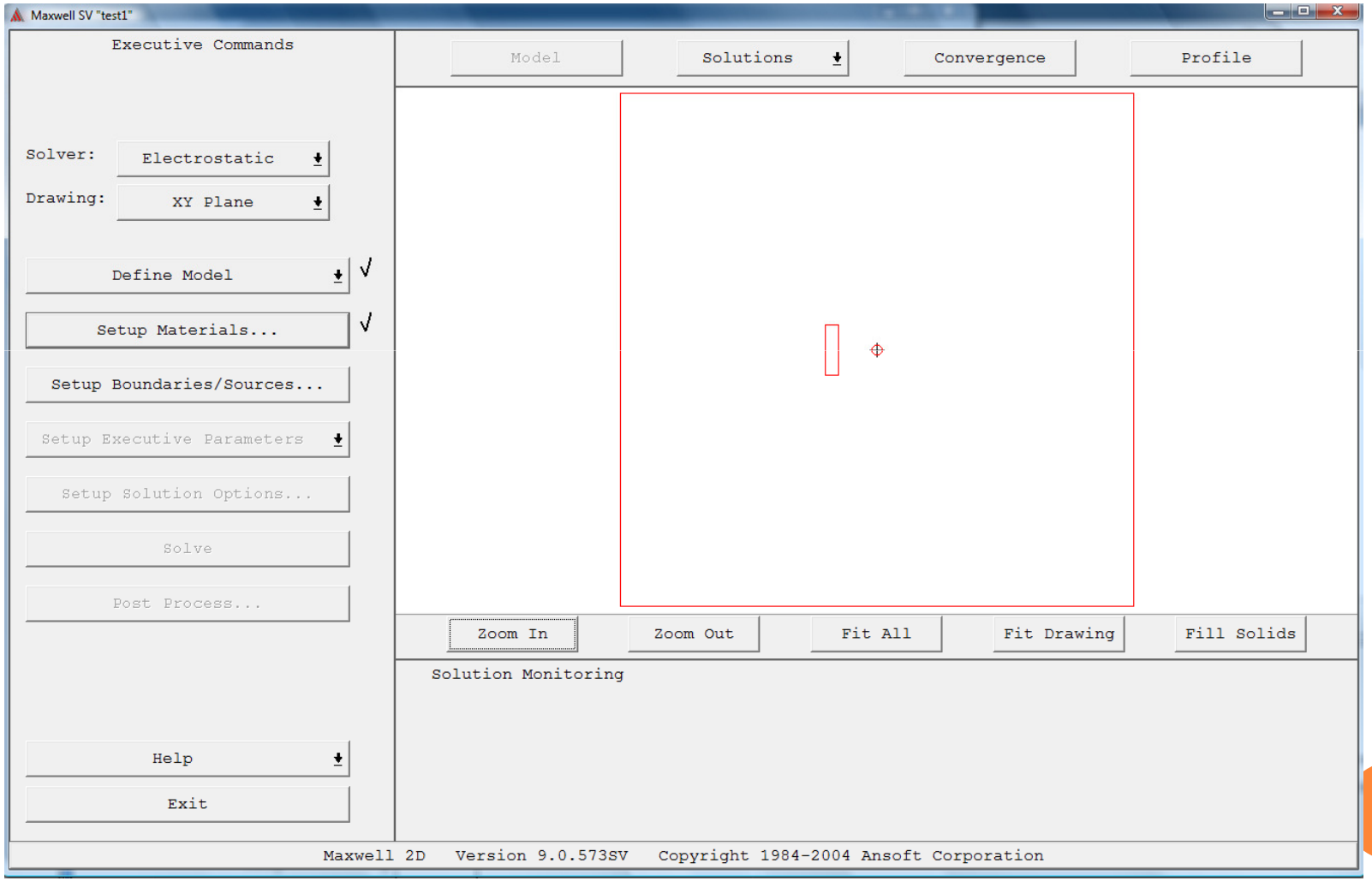
MAXWELL – DEFINICIJA MATERIJALA 2/3



MAXWELL – DEFINICIJA MATERIJALA 3/3

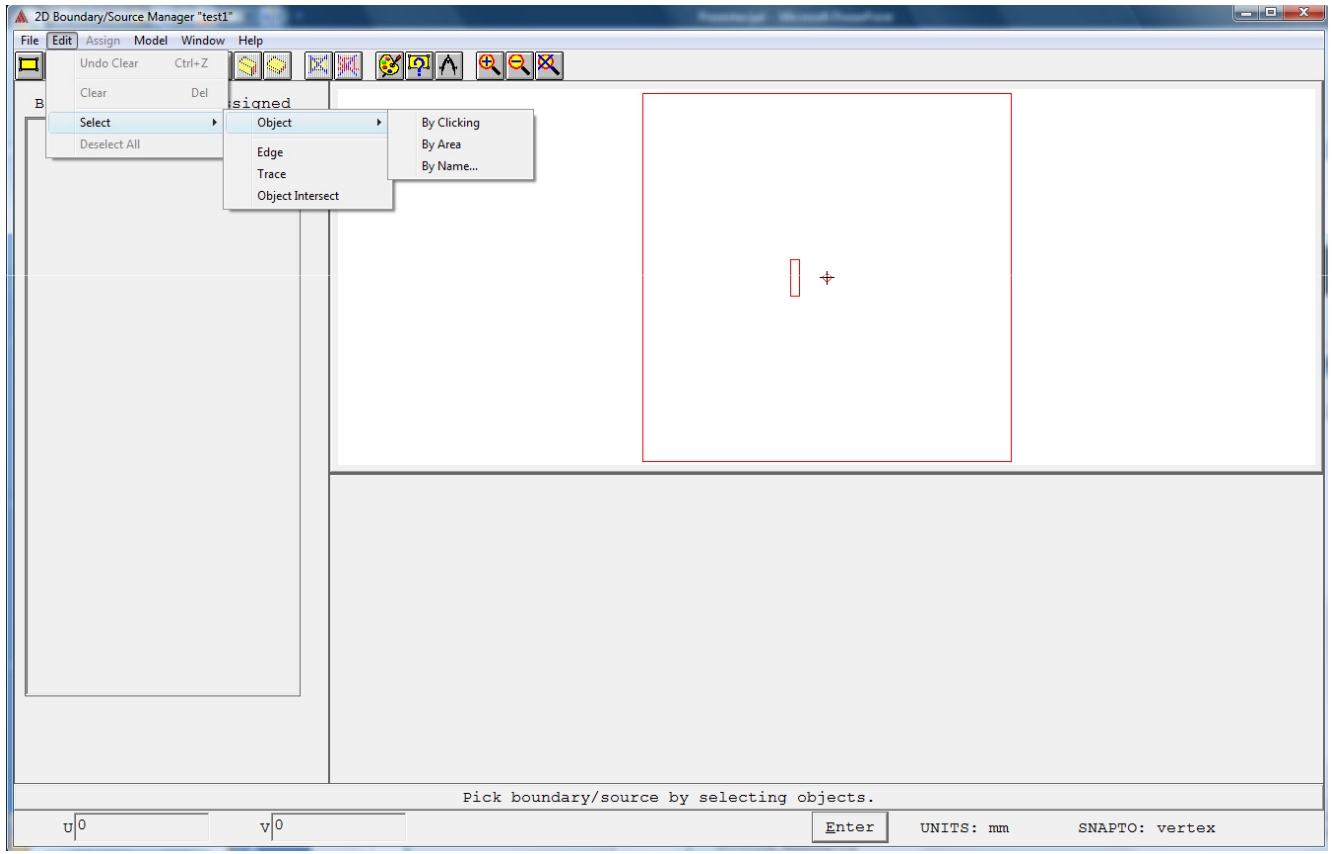


MAXWELL – ZADAVANJE RUBNIH UVJETA I IZVORA



Maxwell – Zadavanje rubnih uvjeta i izvora

- Rubu elementa zadaje se rubni uvjet/izvor



RUBNI UVJETI 1/2

- Balloon – prazni prostor u okolini prostire se u beskonačnost
- Value – propisivanje potencijala na granici
- Symmetry
  - Even (isti predznak na obje strane granice)
  - Odd (suprotan predznak na obje strane granice)
- Resistance – za modeliranje tankih slojeva (problem s generiranjem mreže)

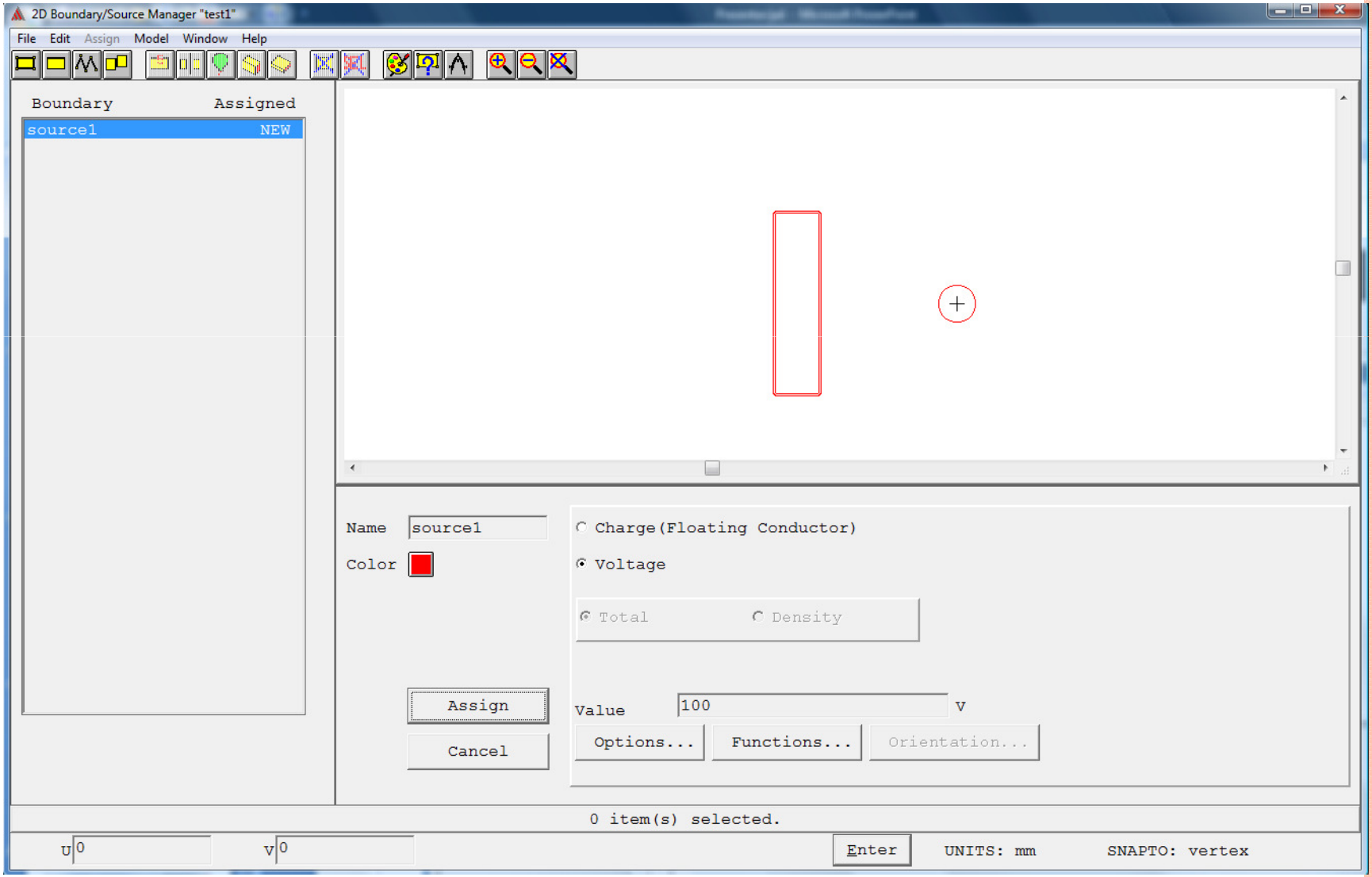
RUBNI UVJETI 2/2

- Impedance – inducirane struje u vodičima s malom dubinom prodiranja
- Master/Slave – polje na slave granici podudara se s poljem na master granici
  - Korisno za geometrijski periodičke strukture – npr. dizajn motora

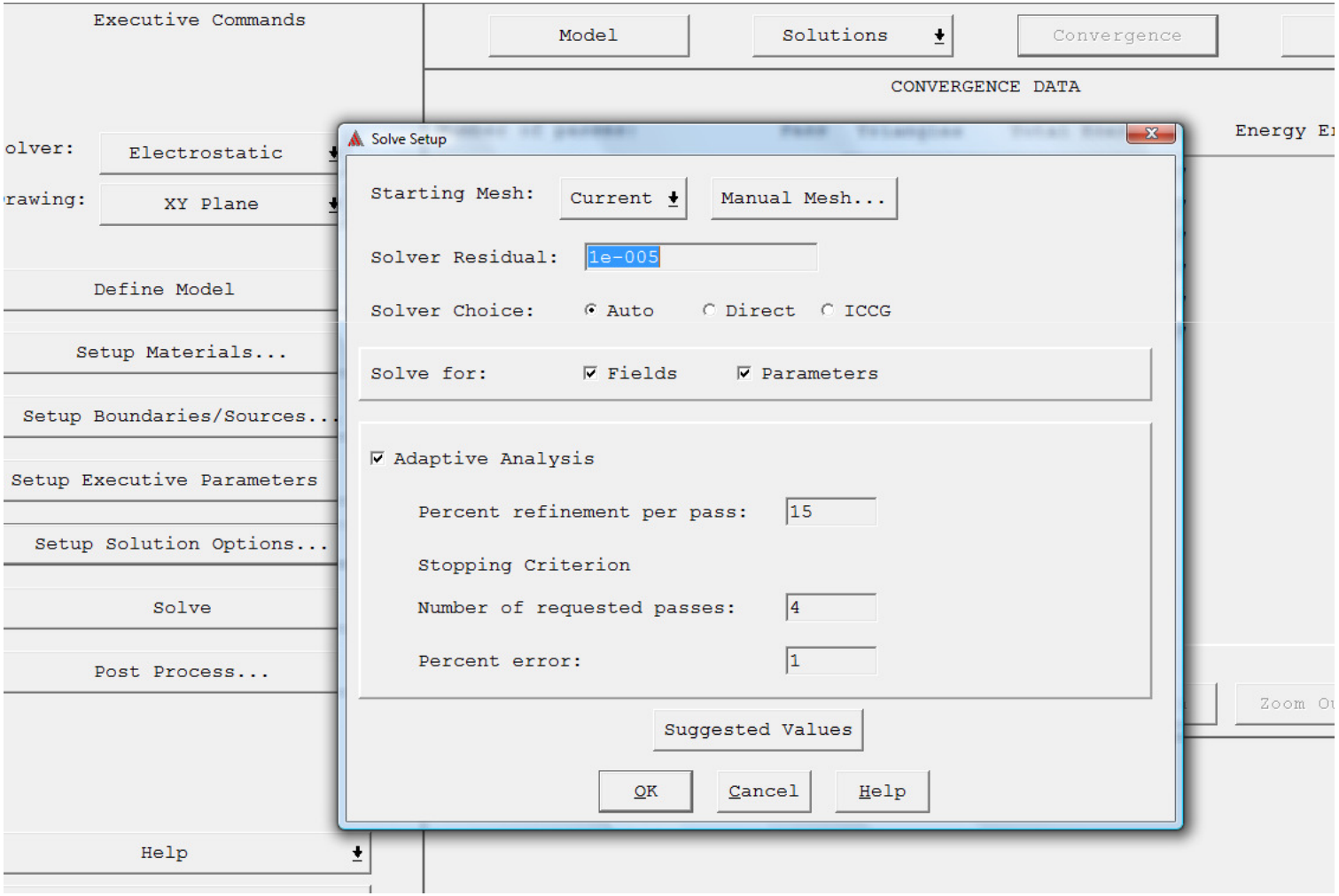
ZADAVANJE IZVORA

- Solid – zadavanje izvora (struje, napona, naboja) označenom objektu
- Sheet – zadavanje izvora (struje, napona, naboja) označenom rubu objekta

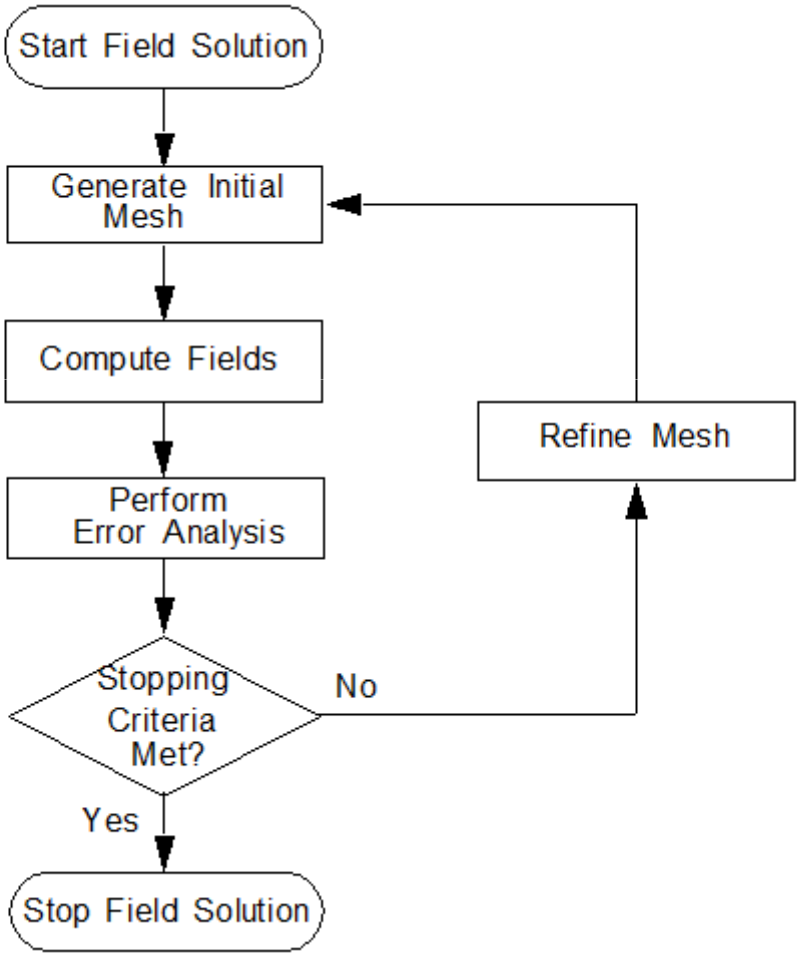
MAXWELL – ZADAVANJE IZVORA I RUBNIH UVJETA



POSTAVKE RJEŠAVANJA PROBLEMA



ADAPTIVNA ANALIZA



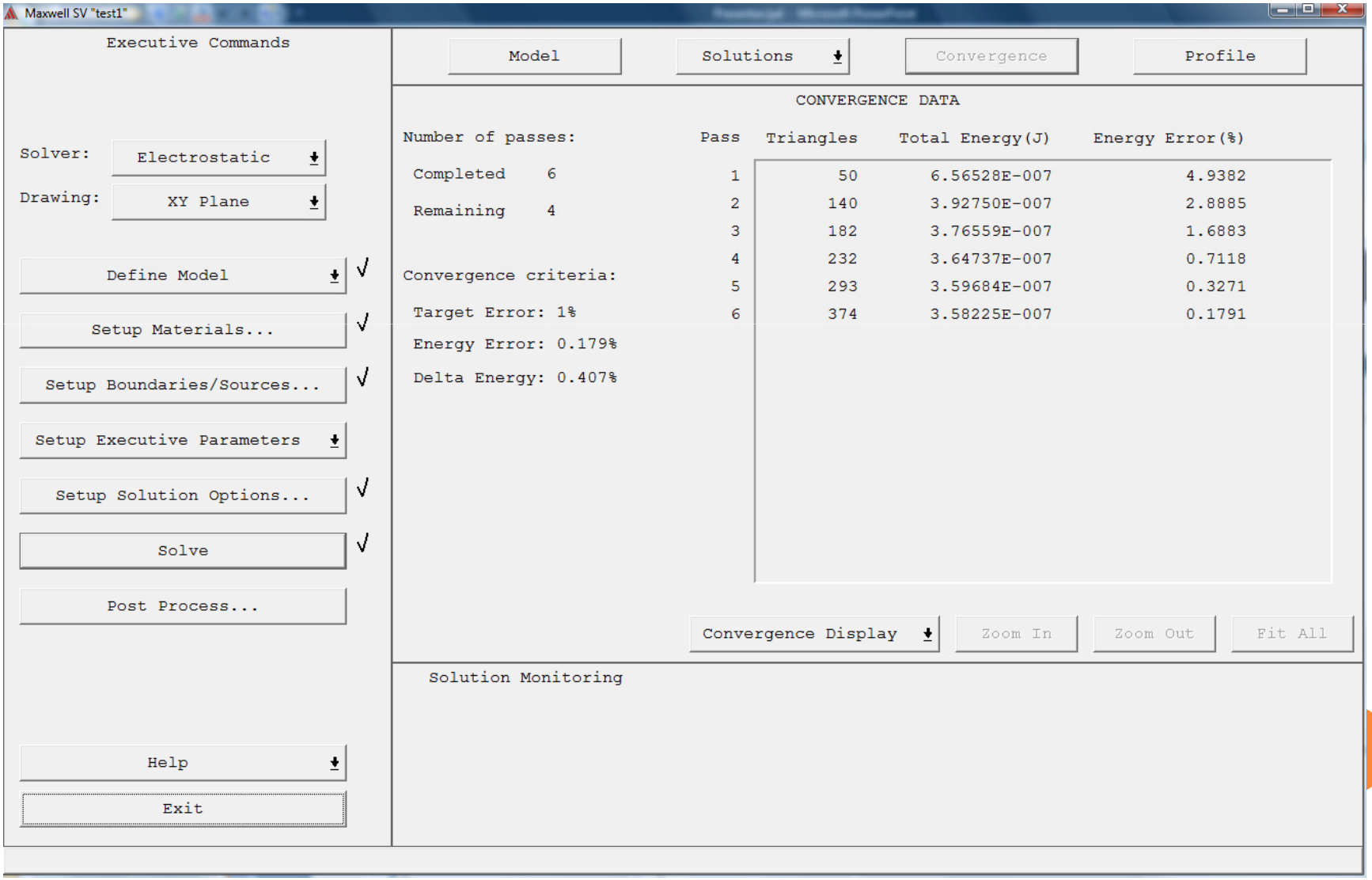
Generiranje mreže

- Iterativni postupak u rješavanju problema
- Current – korištenje zadnje generirane mreže u postupku iteracije
- Manual – ručno postavljanje mreže u željenoj regiji

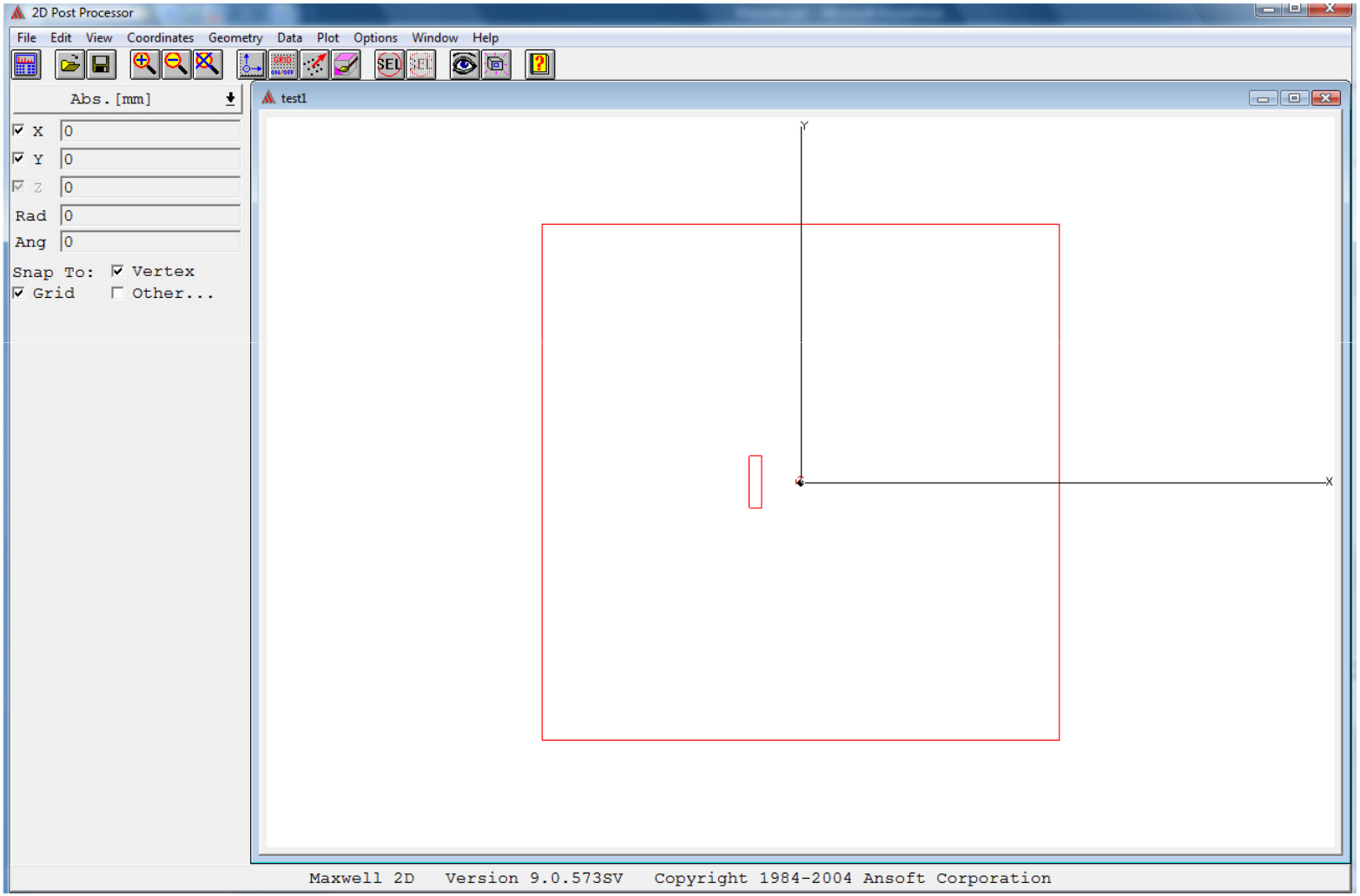
IZBOR NUMERIČKE METODE U PRORAČUNU

- Auto – sustav određuje pristup
- Direct – uvijek konvergentan, ali spor za velike sustave
- ICCG – incomplete conjugate gradient solver,
  - Ponekad nije konvergentan
  - Brži za velike matrice sustava

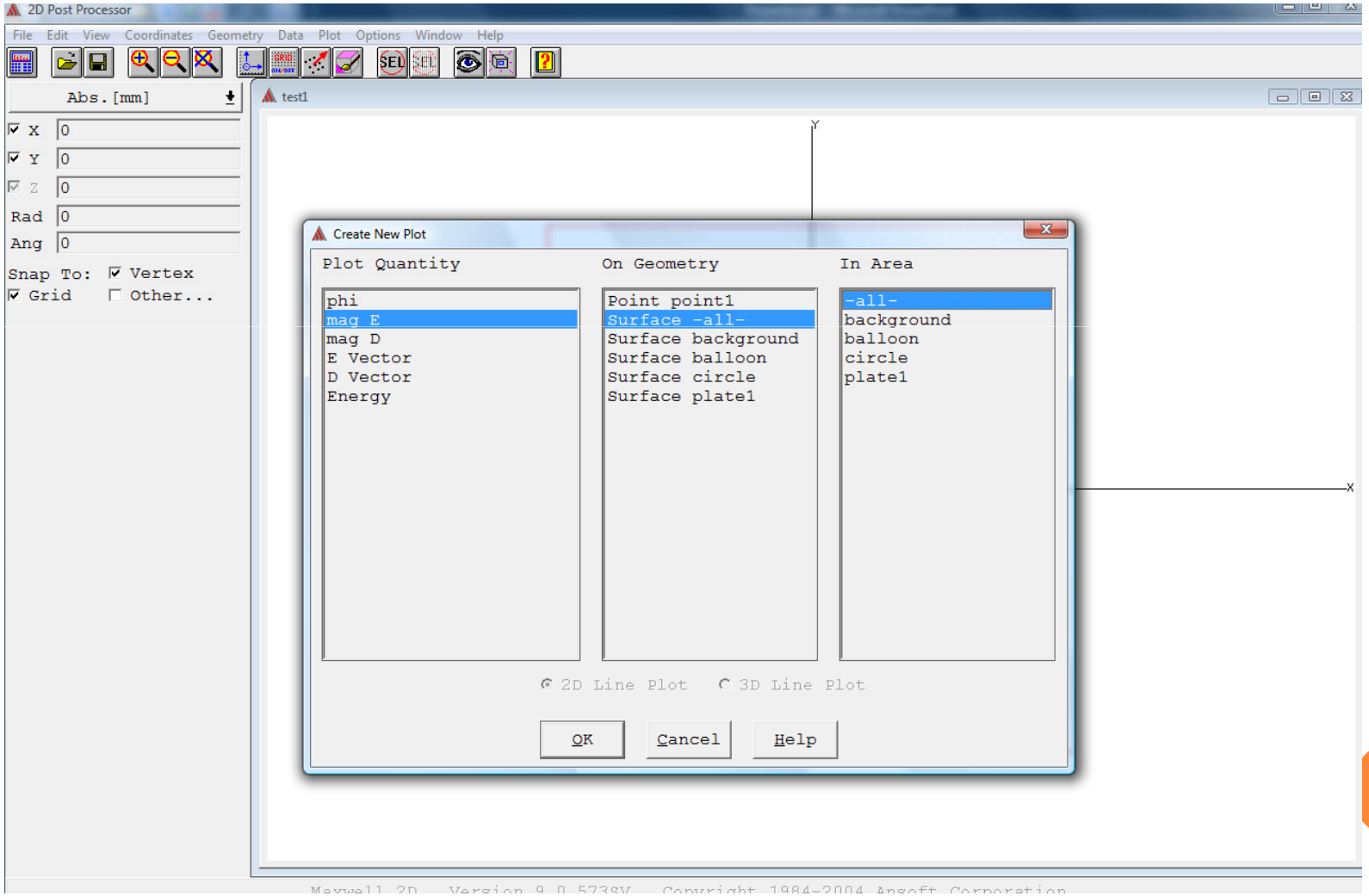
ITERATIVNI POSTUPAK U RJEŠAVANJU PROBLEMA



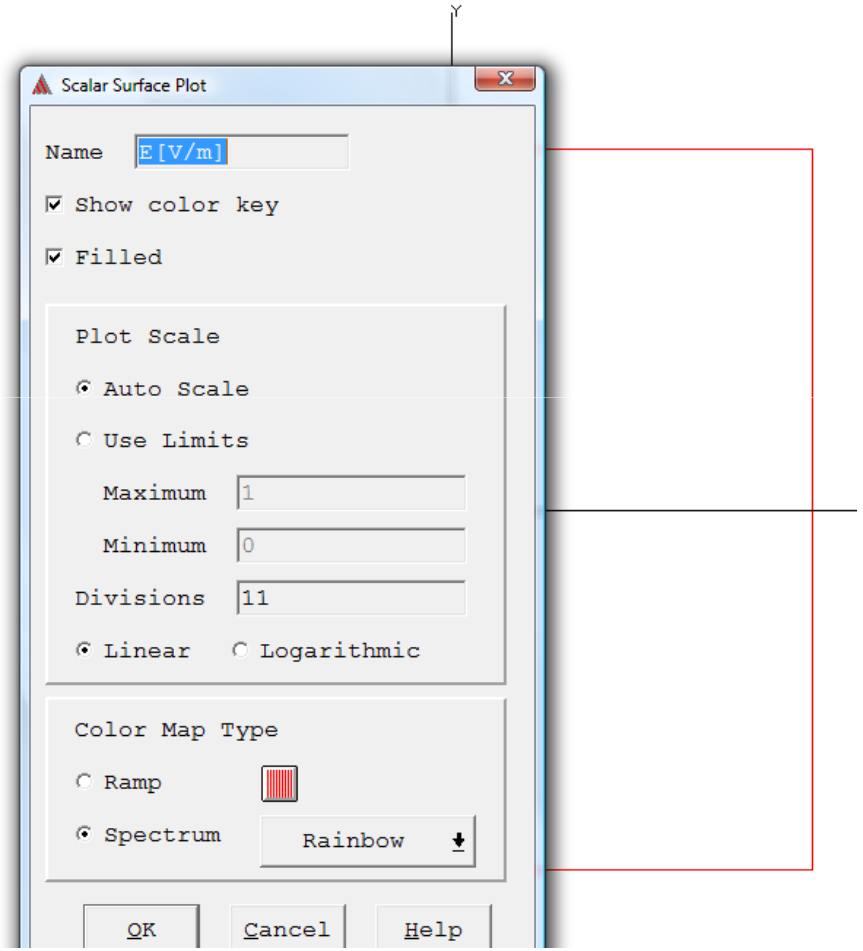
PRIKAZ REZULTATA 1/4



PRIKAZ REZULTATA 2/4

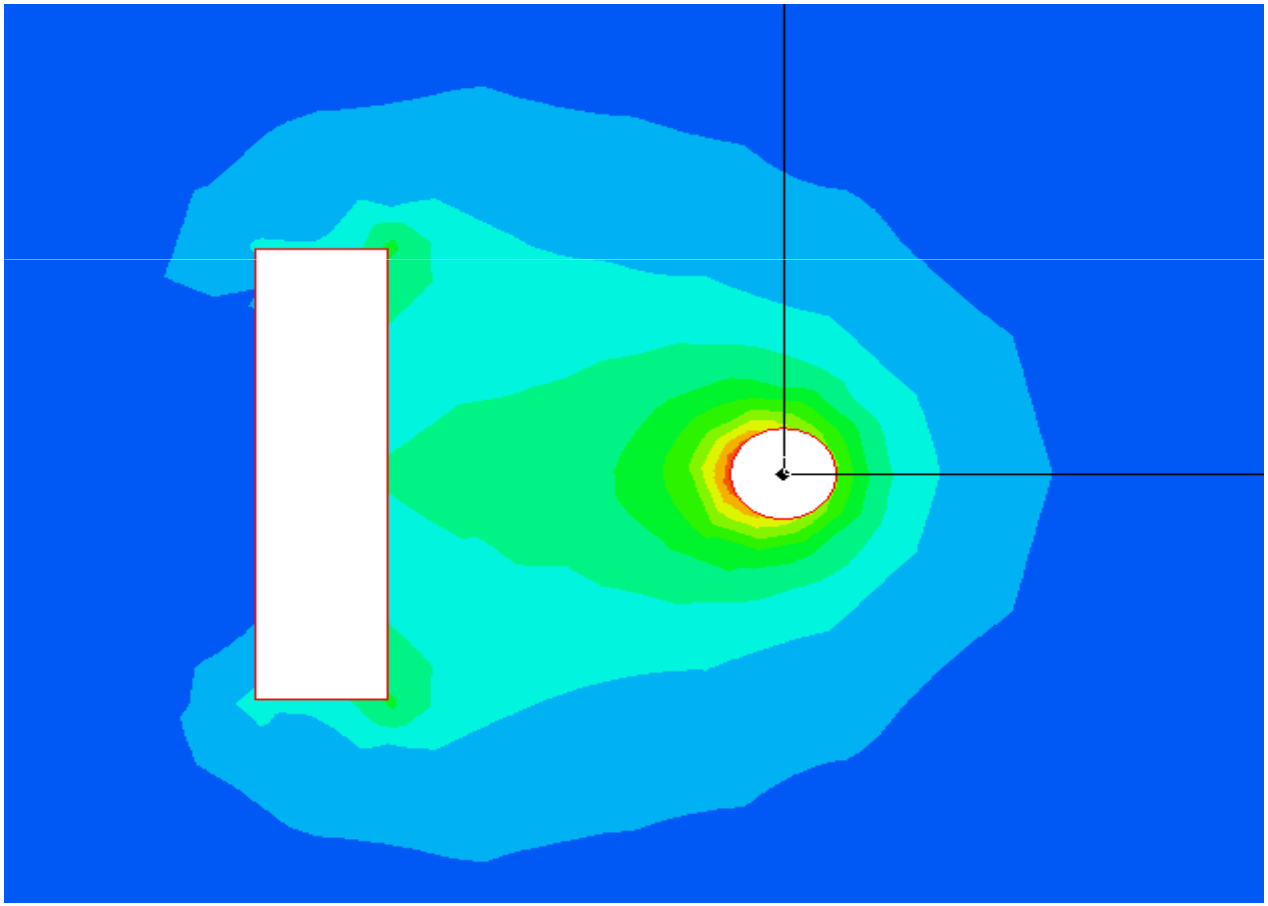


PRIKAZ REZULTATA 3/4

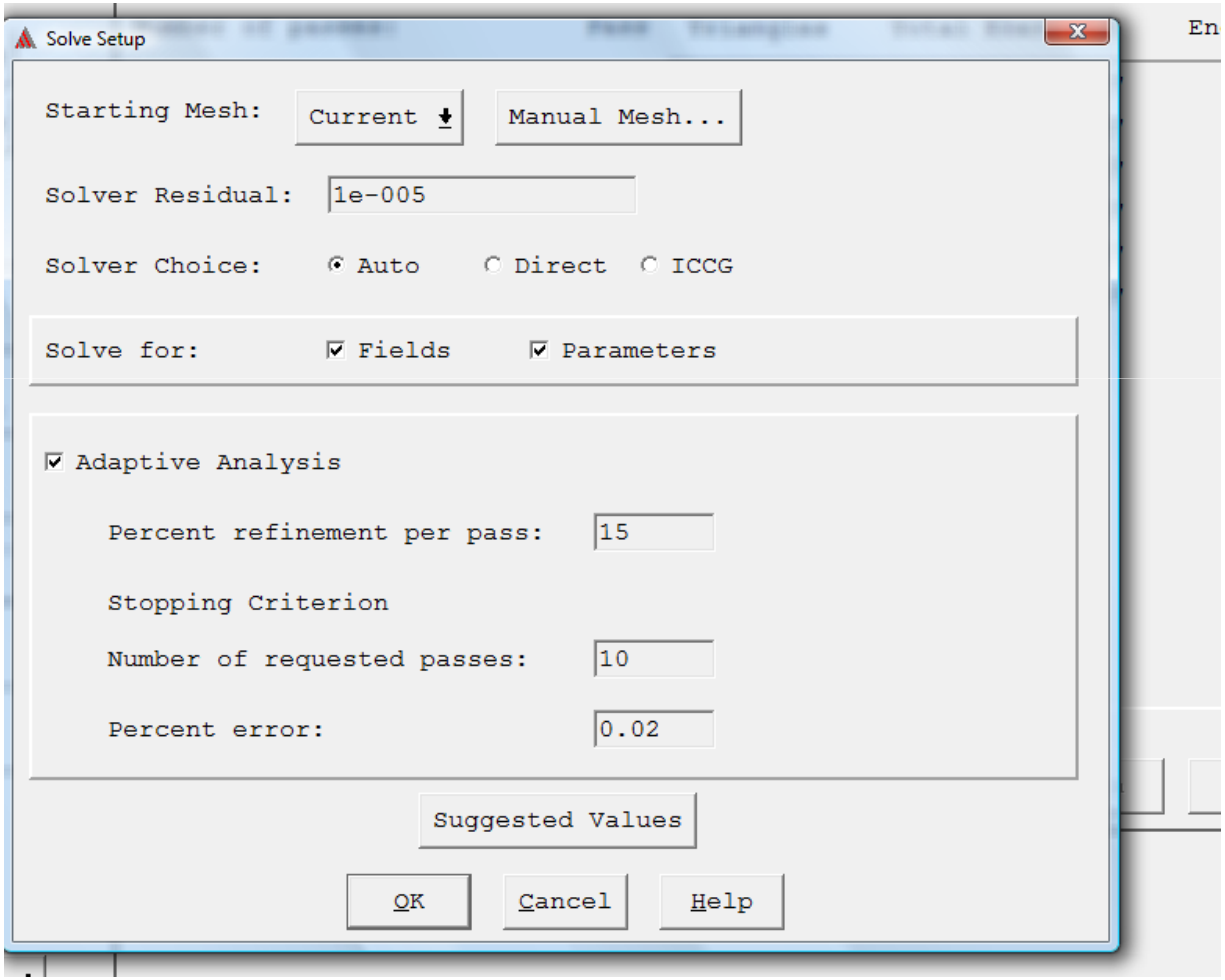


PRIKAZ REZULTATA 4/4

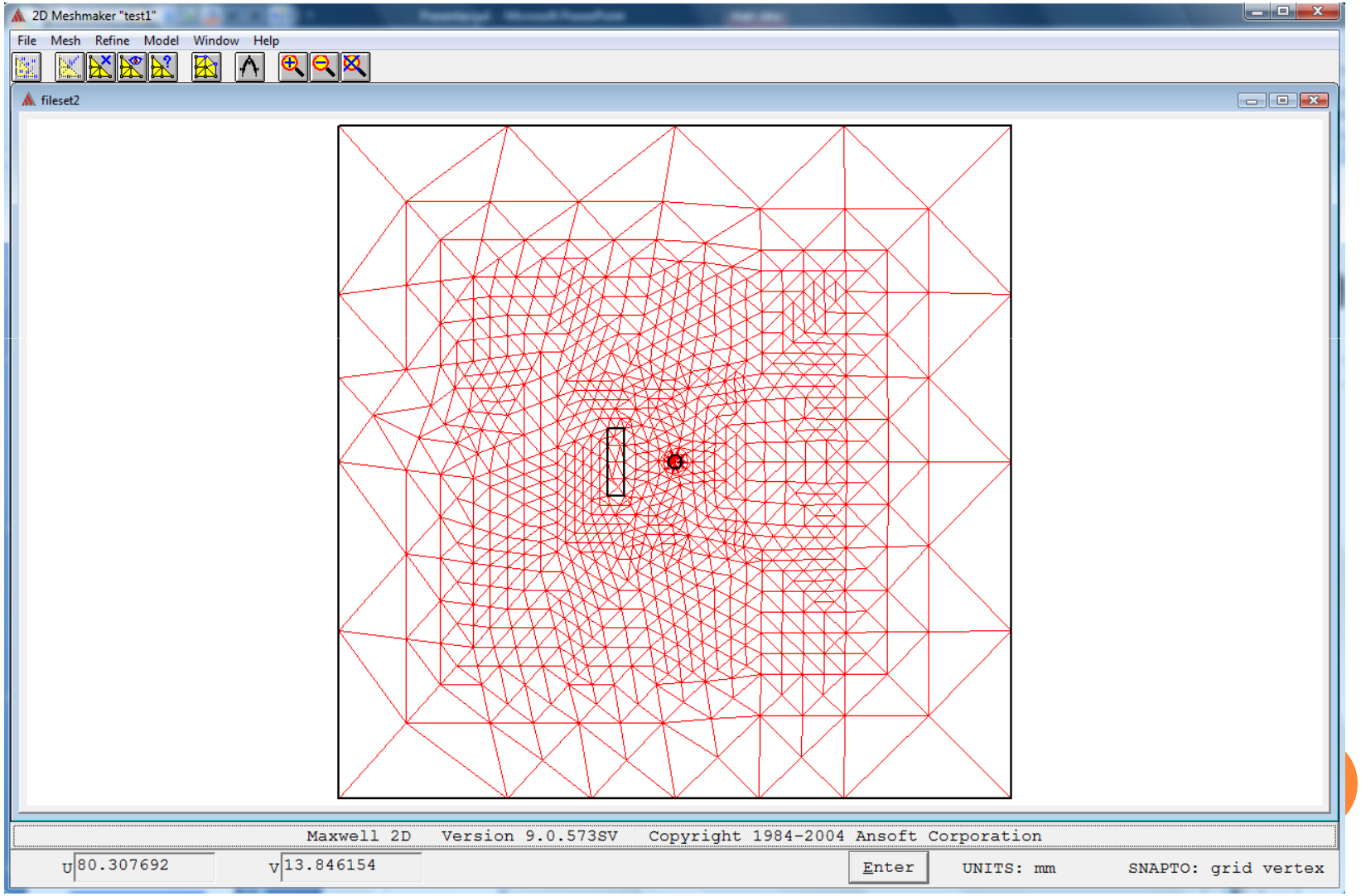
- Vidna nedostatna točnost→prilagodba proračuna



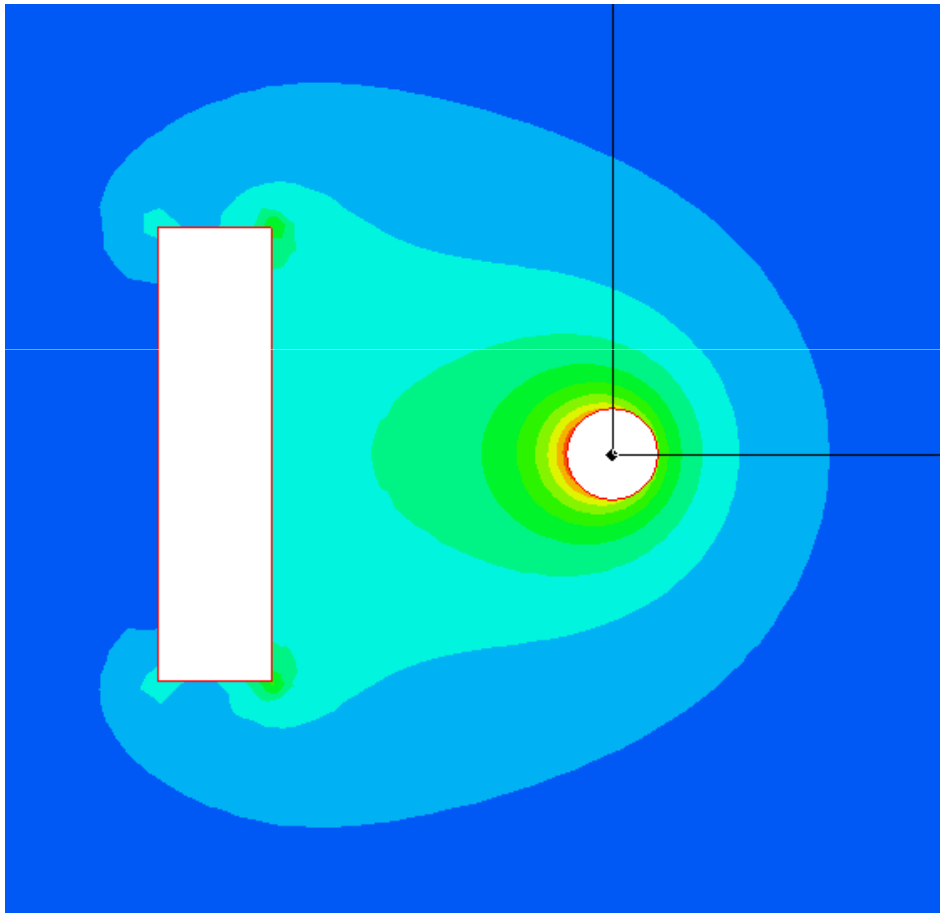
PRILAGODBA ŽELJENOG POSTOTNOG ODSTUPANJA



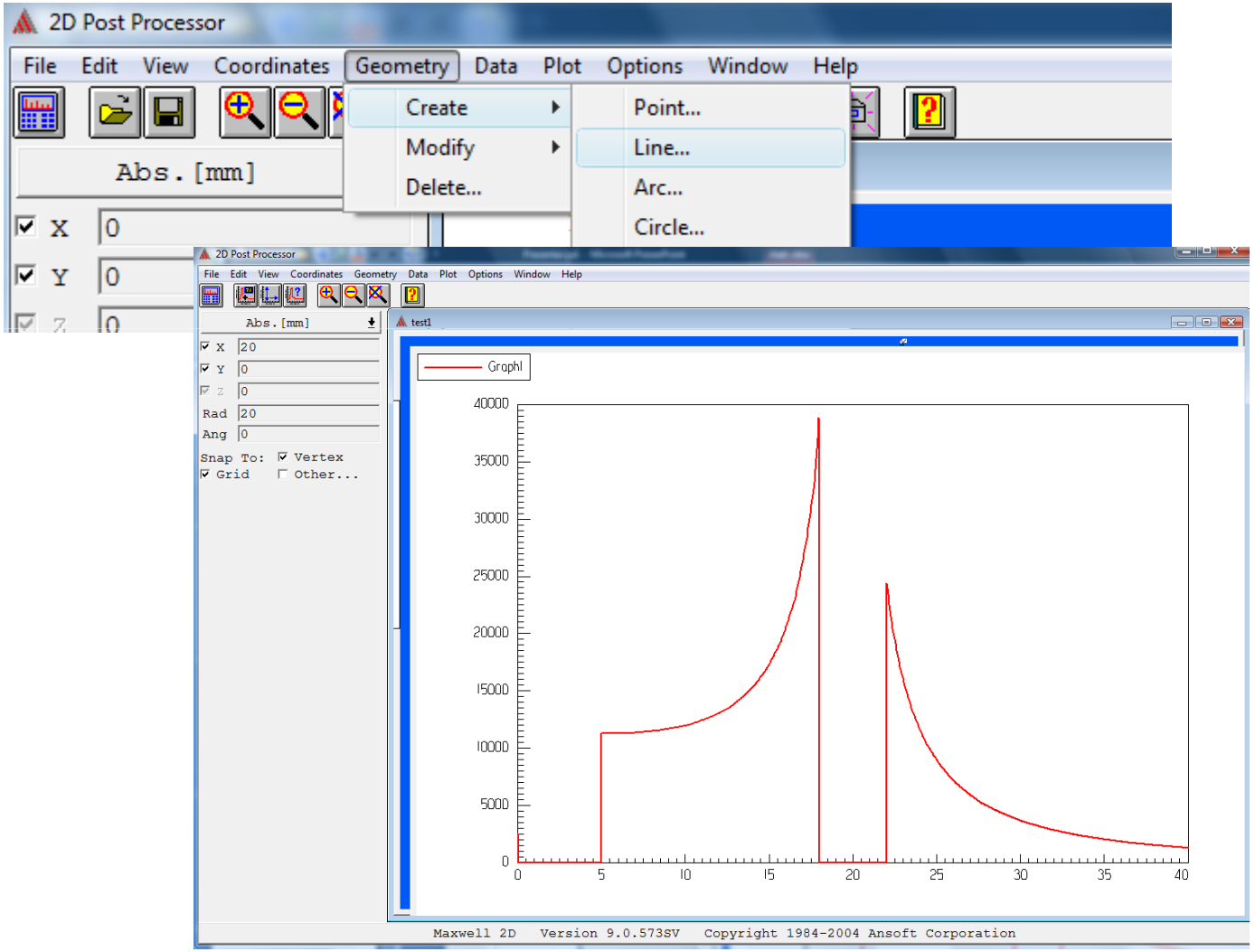
RUČNO GENERIRANJE MREŽE



TOČNIJE RJEŠENJE

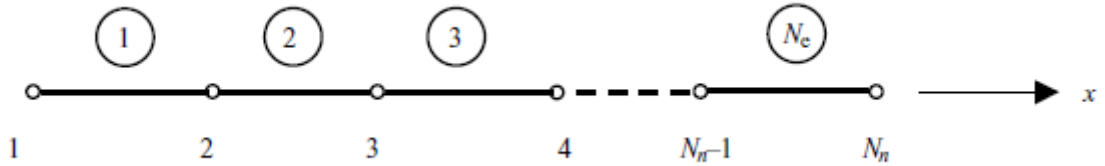


KREIRANJE LINIJE I 2D PLOT PO LINIJI

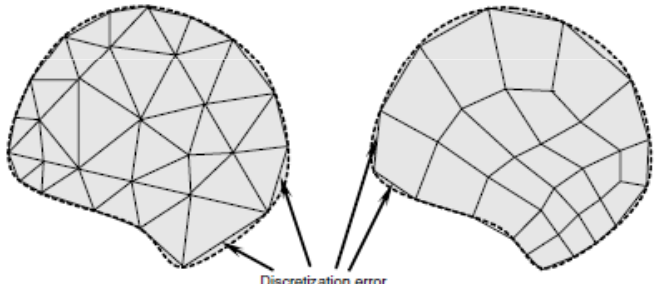


POGLED U STRUKTURU MKE PROGRAMA

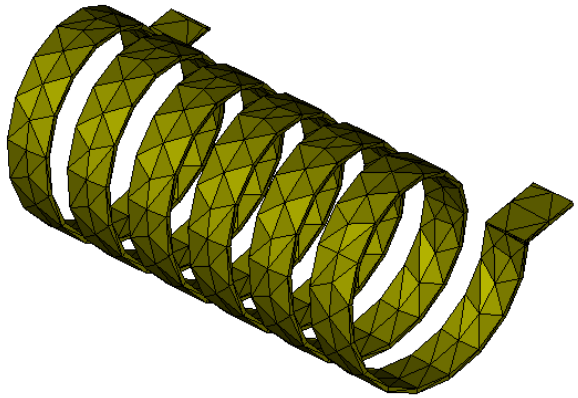
- Diskretizacija domene proračuna konačnim elementima
- 1D



- 2D

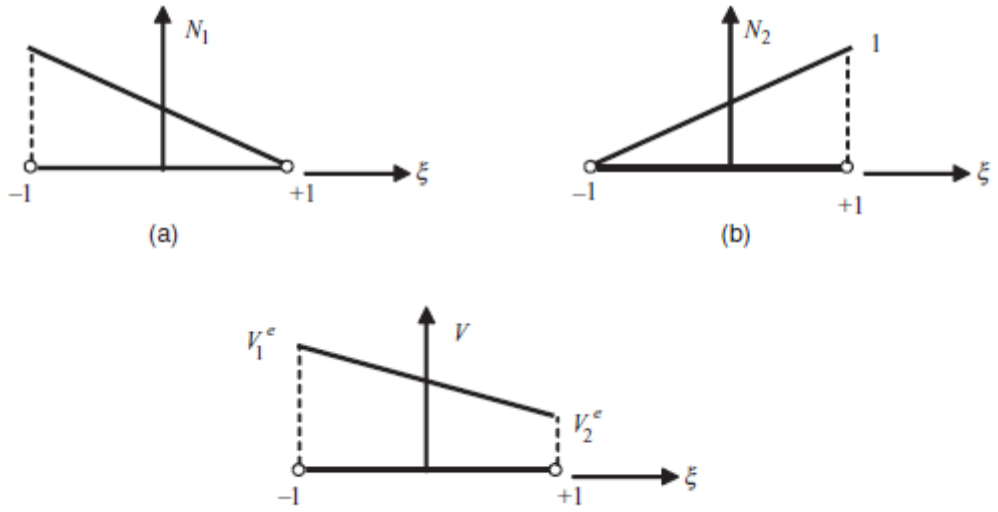


- 3D



IZBOR BAZNIH FUNKCIJA NA ELEMENTU 1D

- Linearne funkcije na 1D elementu



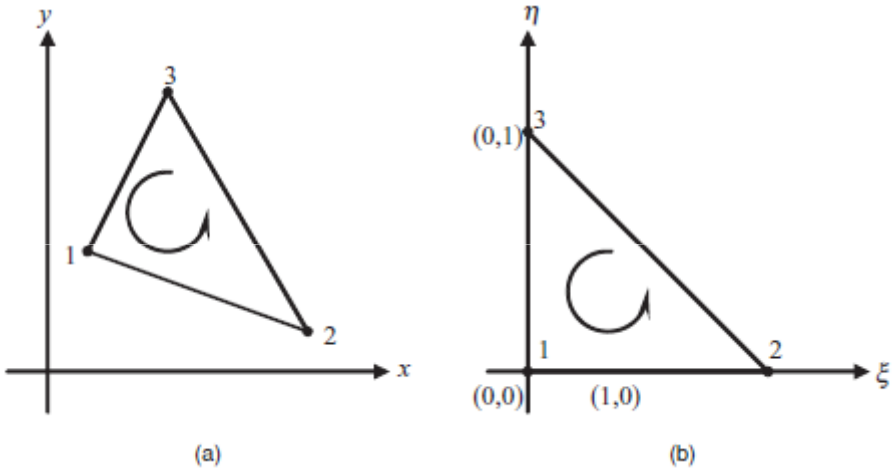
$$V(\xi) = V_1^e N_1(\xi) + V_2^e N_2(\xi)$$

$$N_1(\xi) = \frac{1-\xi}{2} \qquad (-1 \le \xi \le 1)$$

$$N_2(\xi) = \frac{1+\xi}{2}$$

Izbor baznih funkcija na elementu 2D

- Linearne funkcije na 2D elementu



$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$

$$u = u_1^e N_1 + u_2^e N_2 + u_3^e N_3 = \sum_{i=1}^{3} u_i^e N_i$$

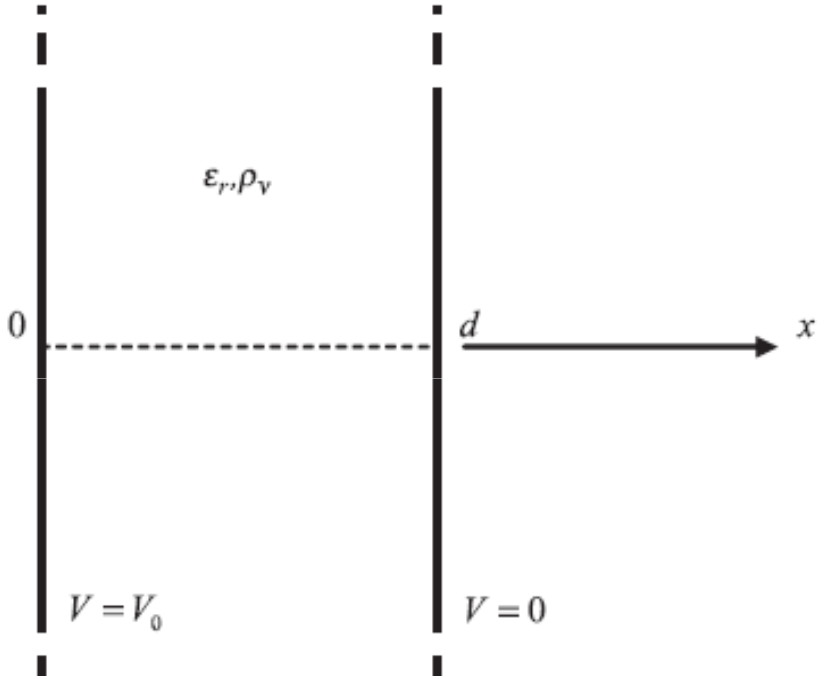
PROMATRAMO POISSONOVU JEDNADŽBU U 1D

- Poissonova jednadžba

$$\nabla(\varepsilon_r \nabla V) = -\frac{\rho_v}{\varepsilon_0}$$

- Rubni uvjeti

$$V(0) = V_0$$
$$V(d) = 0$$



- Analitičko rješenje postoji

$$V(x) = \frac{\rho_0}{2\varepsilon_r \varepsilon_0} x^2 - \left(\frac{\rho_0 d}{2\varepsilon_r \varepsilon_0} + \frac{V_0}{d} \right) x + V_0$$

$$\vec{E}(x) = -\nabla V = -\hat{a}_x \frac{dV(x)}{dx}$$

$$\vec{E}(x) = \hat{a}_x \left[\frac{V_0}{d} + \frac{\rho_0 d}{2\varepsilon_r \varepsilon_0} - \frac{\rho_0 x}{\varepsilon_r \varepsilon_0} \right]$$

GALERKINOV PRISTUP RJEŠAVANJU SUSTAVA

- Postavlja se forma jednadžbe

$$\int_{x_1^e}^{x_2^e} w\left[\frac{d}{dx}\left(\varepsilon^e \frac{dV}{dx}\right) + \rho_v\right] dx = 0$$

- Za težinske funkcije biraju se bazne funkcije
- Parcijalna integracija

$$\int_a^b U\,dV = U\,V|_a^b - \int_a^b V\,dU$$

$$\int_{x_1^e}^{x_2^e} w\frac{d}{dx}\left(\varepsilon^e \frac{dV}{dx}\right) dx = \int_{x_1^e}^{x_2^e} w\,d\left(\varepsilon^e \frac{dV}{dx}\right) = w\,\varepsilon^e \frac{dV}{dx}\bigg|_{x_1^e}^{x_2^e} - \int_{x_1^e}^{x_1^e} \left(\frac{dw}{dx}\right) \varepsilon^e \left(\frac{dV}{dx}\right) dx$$

$$\int_{x_1^e}^{x_2^e} \left(\frac{dw}{dx}\right) \varepsilon^e \left(\frac{dV}{dx}\right) dx - \int_{x_1^e}^{x_2^e} w\,\rho_v\,dx - w\,\varepsilon^e \frac{dV}{dx}\bigg|_{x_1^e}^{x_2^e} = 0$$

Galerkinov pristup rješavanju sustava 2/3

- Konačno

$$D_x^e = -\varepsilon^e \frac{dV}{dx}$$

$$\int_{x_1^e}^{x_2^e} \left(\frac{dw}{dx}\right) \varepsilon^e \left(\frac{dV}{dx}\right) dx - \int_{x_1^e}^{x_2^e} w\,\rho_v\,dx + w\left(x_2^e\right) D_x^e\left(x_2^e\right) - w\left(x_1^e\right) D_x^e\left(x_1^e\right) = 0$$

$$\int_{x_1^e}^{x_2^e} \left(\frac{dN_1}{dx}\right) \varepsilon^e \left(\sum_{j=1}^{2} v_j^e \frac{dN_j}{dx}\right) dx = \int_{x_1^e}^{x_2^e} N_1 \rho_v dx + N_1\left(x_1^e\right) D_x^e\left(x_1^e\right) - N_1\left(x_2^e\right) D_x^e\left(x_2^e\right)$$

$$\int_{x_1^e}^{x_2^e} \left(\frac{dN_2}{dx}\right) \varepsilon^e \left(\sum_{j=1}^{2} v_j^e \frac{dN_j}{dx}\right) dx = \int_{x_1^e}^{x_2^e} N_2 \rho_v dx + N_2\left(x_1^e\right) D_x^e\left(x_1^e\right) - N_2\left(x_2^e\right) D_x^e\left(x_2^e\right)$$

$$\begin{aligned} N_1\left(x_1^e\right) &= 1 \\ N_2\left(x_1^e\right) &= 0 \\ N_1\left(x_2^e\right) &= 0 \\ N_2\left(x_2^e\right) &= 1 \end{aligned} \quad \rightarrow$$

$$\int_{x_1^e}^{x_2^e} \left(\frac{dN_1}{dx}\right) \varepsilon^e \left(\sum_{j=1}^{2} v_j^e \frac{dN_j}{dx}\right) dx = \int_{x_1^e}^{x_2^e} N_1 \rho_v dx + D_x^e\left(x_1^e\right)$$

$$\int_{x_1^e}^{x_2^e} \left(\frac{dN_2}{dx}\right) \varepsilon^e \left(\sum_{j=1}^{2} v_j^e \frac{dN_j}{dx}\right) dx = \int_{x_1^e}^{x_2^e} N_2 \rho_v dx - D_x^e\left(x_2^e\right)$$

GALERKINOV PRISTUP RJEŠAVANJU SUSTAVA 3/3

- Formiranje matričnog zapisa

$$\begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix} \begin{Bmatrix} v_1^e \\ v_2^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix} + \begin{Bmatrix} D_1^e \\ -D_2^e \end{Bmatrix}$$

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left(\frac{dN_i}{dx} \right) \varepsilon^e \left(\frac{dN_j}{dx} \right) dx$$

$$f_i^e = \int_{x_1^e}^{x_2^e} N_i \rho_v dx$$

Zapis za jedan linearni element za Poissonovu jednadžbu

- Izvedeni izrazi prema prethodnim formulama

$$\frac{dN_1}{d\xi} = -\frac{1}{2} \qquad \frac{dN_2}{d\xi} = +\frac{1}{2}$$

$$K^e = \frac{\varepsilon^e}{l^e}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \qquad \mathbf{f}^e = -\frac{l^e \rho_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\frac{\varepsilon^e}{l^e}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\begin{Bmatrix} v_1^e \\ v_2^e \end{Bmatrix} = -\frac{l^e \rho_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} D_1^e \\ -D_2^e \end{Bmatrix}$$

Sastavljanje sustava jednadžbi 1/4

- Općenito

$$K_{11}^{e} v_1^{e} + K_{12}^{e} v_2^{e} = f_1^{e} + D_1^{e}$$
$$K_{21}^{e} v_1^{e} + K_{22}^{e} v_2^{e} = f_2^{e} - D_2^{e}$$

- 1

$$K_{11}^{(1)} v_1^{(1)} + K_{12}^{(1)} v_2^{(1)} = f_1^{(1)} + D_1^{(1)}$$
$$K_{21}^{(1)} v_1^{(1)} + K_{22}^{(1)} v_2^{(1)} = f_2^{(1)} - D_2^{(1)}$$

- 2

$$K_{11}^{(2)} v_1^{(2)} + K_{12}^{(2)} v_2^{(2)} = f_1^{(2)} + D_1^{(2)}$$
$$K_{21}^{(2)} v_1^{(2)} + K_{22}^{(2)} v_2^{(2)} = f_2^{(2)} - D_2^{(2)}$$

- 3

$$K_{11}^{(3)} v_1^{(3)} + K_{12}^{(3)} v_2^{(3)} = f_1^{(3)} + D_1^{(3)}$$
$$K_{21}^{(3)} v_1^{(3)} + K_{22}^{(3)} v_2^{(3)} = f_2^{(3)} - D_2^{(3)}$$

- .....

Sastavljanje sustava jednadžbi 2/4

- Vrijedi

$$v_2^e = v_1^{e+1}$$

$$v_2^{(1)} = v_1^{(2)} = V_2$$
$$v_2^{(2)} = v_1^{(3)} = V_3$$
$$v_2^{(3)} = v_1^{(4)} = V_4$$

$$K_{11}^{(1)} V_1 + K_{12}^{(1)} V_2 = f_1^{(1)} + D_1^{(1)}$$
$$K_{21}^{(1)} V_1 + K_{22}^{(1)} V_2 = f_2^{(1)} - D_2^{(1)}$$

$$K_{11}^{(2)} V_2 + K_{12}^{(2)} V_3 = f_1^{(2)} + D_1^{(2)}$$
$$K_{21}^{(2)} V_2 + K_{22}^{(2)} V_3 = f_2^{(2)} - D_2^{(2)}$$

$$K_{11}^{(3)} V_3 + K_{12}^{(3)} V_4 = f_1^{(3)} + D_1^{(3)}$$
$$K_{21}^{(3)} V_3 + K_{22}^{(3)} V_4 = f_2^{(3)} - D_2^{(3)}$$

SASTAVLJANJE SUSTAVA JEDNADŽBI 3/4

- Matrični zapis – rijetko popunjena matrica MKE

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & & & \\ K_{21}^{(1)} & \left(K_{22}^{(1)} + K_{11}^{(2)}\right) & K_{12}^{(2)} & & & \\ & K_{21}^{(2)} & \left(K_{22}^{(2)} + K_{11}^{(3)}\right) & K_{12}^{(3)} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & K_{21}^{(N_e-1)} & \left(K_{22}^{(N_e-1)} + K_{11}^{(N_e)}\right) & K_{12}^{(N_e)} & \\ & & & K_{21}^{(N_e)} & K_{22}^{(N_e)} & \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{N_e} \\ V_{N_e+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ \vdots \\ f_2^{(N_e-1)} + f_1^{(N_e)} \\ f_2^{(N_e)} \end{Bmatrix} + \begin{Bmatrix} D_1^{(1)} \\ -D_2^{(1)} + D_1^{(2)} \\ -D_2^{(2)} + D_1^{(3)} \\ \vdots \\ -D_2^{(N_e-1)} + D_1^{(N_e)} \\ -D_2^{(N_e)} \end{Bmatrix}$$

Sastavljanje rubnih uvjeta – Dirichletovi uvjeti prema testnom primjeru

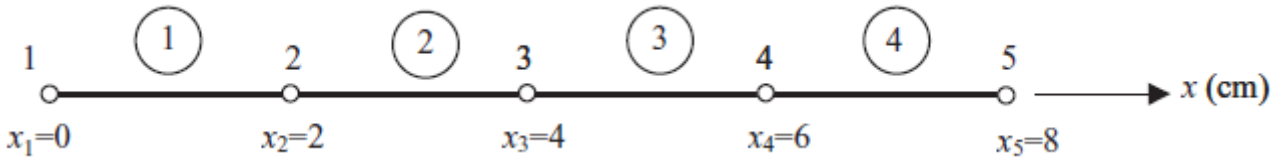
- Propisan potencijal na granici – redukcija dimenzije

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1N} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2N} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N1} & K_{N2} & K_{N3} & \cdots & K_{NN} \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_N \end{Bmatrix} \quad \longleftarrow \quad V_1 = V_0$$

$$\begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2N} \\ K_{32} & K_{33} & \cdots & K_{3N} \\ \vdots & \vdots & \cdots & \vdots \\ K_{N2} & K_{N3} & \cdots & K_{NN} \end{bmatrix} \begin{Bmatrix} V_2 \\ V_3 \\ \vdots \\ V_N \end{Bmatrix} = \begin{Bmatrix} b_2 - K_{21}V_0 \\ b_3 - K_{31}V_0 \\ \vdots \\ b_N - K_{N1}V_0 \end{Bmatrix}$$

PRIMJER PRORAČUNA 1

- Zadani elementi duljine 2cm



$\varepsilon_r = 1$
$V_0 = 1\ \text{V}$
$d = 8\ \text{cm}$
$\rho_0 = 10^{-8}\ \text{C/m}^3$

- Treba izračunati raspodjelu potencijala po elementima

PRIMJER PRORAČUNA 2

- Izračun elemenata za formulaciju sustava

$$K^e = \frac{8.85 \times 10^{-12}}{2 \times 10^{-2}} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} = 4.425 \times 10^{-10} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

$$\mathbf{f^e} = -\frac{2 \times 10^{-2} \times 10^{-8}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -10^{-10} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$4.425 \times 10^{-10} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = -10^{-10} \begin{Bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{Bmatrix}$$

PRIMJER PRORAČUNA 3

- Postavljanje rubnih uvjeta za napon na početku i kraju V_1 =1, V_5=0

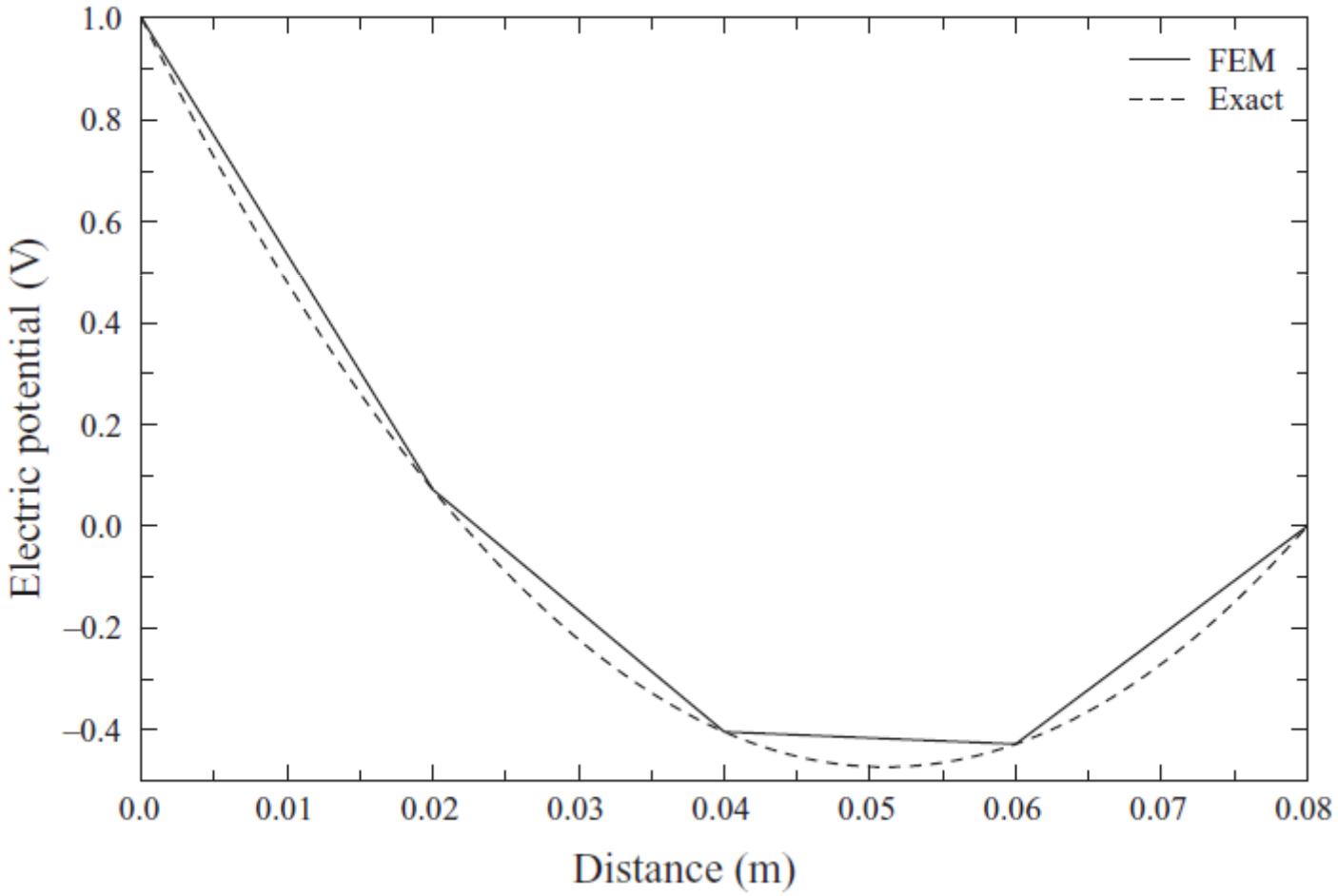
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} V_2 \\ V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} 0.5480226 \\ -0.4519774 \\ -0.4519774 \end{Bmatrix}$$

V_1 =1, V_2 =0,07, V_3 = -0,4, V_4 = -0,43, V_5=0

$$V(x) = V_1^e \left(\frac{x_2^e - x}{x_2^e - x_1^e} \right) + V_2^e \left(\frac{x - x_1^e}{x_2^e - x_1^e} \right)$$

PRIMJER PRORAČUNA 4

- Usporedba s analitičkim izrazom



ZAKLJUČAK

- Pokazan primjer proračuna metodom konačnih elemenata sa svim koracima
  - profesionalnim programom Maxwell
  - na jednostavnom modelu s potpunom matematičkom formulacijom
- MKE
  - robusna metoda
  - široko implementirana u programskim paketima
  - razvoj u zreloj fazi – implementacije na spregnutim proračunima, optimizacije
  - Solveri se izvode za paralelne arhitekture