Rješenja 1. međuspita iz Matematike 3E i 3R

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1. (**2 boda**)
a) (**1 bod**) Za funkcije $f, g : [a, b] \to \mathbb{R}$ kažemo da su ortogonalne na intervalu $[a, b]$ ako vrijedi $\int_a^b f(x)g(x)\,dx = 0$.

b) (**1 bod**) $\int_{-\pi}^{\pi} \sin mx\, \sin nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x)dx =$
$\frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x|_{-\pi}^{\pi} - \frac{1}{m+n}\sin(m+n)x|_{-\pi}^{\pi}\right] = 0$

2. (**3 boda**)
a) (**2 boda**) $f(x) = \frac{\pi}{4} - \frac{x}{2}$ na intervalu $\langle 0, \pi\rangle$ u red po sinus funkcijama $\Rightarrow$ neparna funkcija $\Rightarrow a_n = 0$
$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\pi} (\frac{\pi}{4} - \frac{x}{2})\sin nx\, dx =$
$= -\frac{1}{2n}\cos nx|_0^{\pi} + \frac{1}{n\pi}x\, \cos nx|_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi} \cos nx\, dx = \ldots = \frac{\cos \pi n + 1}{2n}$
za n neparan je $b_n = 0$, a za n paran je $b_n = \frac{1}{n}$ te je Fourierov red
$f(x) = \sum_{n\geq 1} \frac{1}{2n}\sin(2nx)$

b) (**1 bod**) $S(3\pi) = \frac{1}{2}(f(3\pi + 0) + f(3\pi - 0)) = 0.$

3. (**3 boda**) funkcija $f(x)$ je parna $\Rightarrow B(\lambda) = 0$
$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\lambda\xi\, d\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} 2\cos\frac{\xi}{2}\cos\lambda\xi\, d\xi = \frac{2}{\pi}\int_0^{\pi}(\cos\xi(\lambda + \frac{1}{2}) + \cos\xi(\lambda - \frac{1}{2})\, d\xi = \frac{2}{\pi}(\frac{2}{2\lambda+1}\sin\xi(\lambda + \frac{1}{2})|_0^{\pi} + \frac{2}{2\lambda-1}\sin\xi(\lambda - \frac{1}{2})|_0^{\pi}) = \frac{4}{\pi}(\frac{\sin\pi(\lambda+\frac{1}{2})}{2\lambda+1} + \frac{\sin\pi(\lambda-\frac{1}{2})}{2\lambda-1}) = \frac{4}{\pi}(\frac{\cos\pi\lambda}{2\lambda+1} - \frac{\cos\pi\lambda}{2\lambda-1}) = \frac{8}{\pi}\frac{\cos\pi\lambda}{1-4\lambda^2}$, $\lambda \neq \pm\frac{1}{2}$
Fourierov integral $\quad f(x) = \frac{8}{\pi}\int_0^{\infty}\frac{\cos\pi\lambda}{1-4\lambda^2}\cos\lambda x d\lambda$
$\int_0^{\infty}\frac{\cos\pi t}{1-4t^2}dt = \frac{\pi}{8}f(0) = \frac{\pi}{4}.$

4. (**2 boda**)
a) (**1 bod**) $\mathcal{L}(f(t)) = F(s)$
$\mathcal{L}(e^{-\alpha t}f(t)) = \int_0^{\infty} e^{-st}e^{-\alpha t}f(t)dt = \int_0^{\infty} e^{-(s+\alpha)t}f(t)dt = F(s+\alpha)$

b) (**1 bod**) $\mathcal{L}(f'(t)) = \int_0^{\infty} e^{-st}f'(t)dt = (PI) = e^{-st}f(t)\Big|_0^{\infty} + s\int_0^{\infty} e^{-st}f(t)dt = sF(s) + \lim_{t\to\infty} e^{-st}f(t) - f(0) = sF(s) - f(0)$

5. (**2 boda**) $f(t) = (t-2)^3\, e^{-t}\, u(t-2)$
$t^3 \circ\!-\!\bullet\ \frac{3!}{s^4}$,
$(t-2)^3 u(t-2) \circ\!-\!\bullet\ \frac{6}{s^4}e^{-2s}$
$f(t) = (t-2)^3\, e^{-t}\, u(t-2) \circ\!-\!\bullet\ \frac{6}{(s+1)^4}e^{-2(s+1)}$

6. (**2 boda**) $f(t) = t^{100} \circ\!-\!\bullet\ \frac{100!}{s^{101}} = F(s)$
$\int_0^{\infty} e^{-t}\, t^{100} dt = F(1) = 100!.$

7. (**3 boda**) $y(t) = \sin t + \int_0^t \tau\, y(t-\tau)d\tau$
$\int_0^t \tau\, y(t-\tau)d\tau = t * y(t) \circ\!-\!\bullet \frac{1}{s^2}Y(s)$
$Y(s) = \frac{1}{1+s^2} + \frac{1}{s^2}Y(s) \Rightarrow Y(s) = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1} = \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1}$
$y(t) = (\frac{1}{2}\sin t + \frac{1}{4}e^t - \frac{1}{4}e^{-t})u(t)$

8. (**3 boda**) $e(t) = \frac{t}{2}(u(t) - u(t-2)) + u(t-2) = \frac{1}{2}tu(t) - \frac{1}{2}(t-2)u(t-2)$
$E(s) = \frac{1}{2s^2} - \frac{1}{2s^2}e^{-2s}$
$Z(s) = \frac{1\frac{1}{s}}{1+\frac{1}{s}} = \frac{1}{s+1}$
$I(s) = \frac{E(s)}{Z(s)} = \frac{s+1}{2s^2}(1-e^{-2s}) = \frac{1}{2s} + \frac{1}{2s^2} - \frac{1}{2s}e^{-2s} - \frac{1}{2s^2}e^{-2s}$
$i(t) = \frac{1}{2}(u(t) + tu(t) - u(t-2) - (t-2)u(t-2)) = \frac{1}{2}[(1+t)u(t) + (1-t)u(t-2)]$